An alternative proof for the double Pareto distribution

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Abstract

This note provides an alternative proof for the double Pareto distribution that appears in Reed 2001 and Arkolakis 2008. The proof is computing cross-generational productivity distribution by weighting the distribution of each generation by its relative population.

1 Introduction

Given the assumptions stated in (Arkolakis 2007) who in turn follows (Reed 2001) we will show an alternative way to prove that the stationary distibution arising from the assumptions of continuous entry at a certain rate and geometric Brownian motion is the double-Pareto. To do that we will look directly at the distribution of sizes of the cross-section of firms with different productivities which involves properly weighting the distributions of firms of different generations.

2 The details of the proof

We assume that all ideas initiate with productivity $\bar{z}$. The logarithm of a geometric Brownian motion follows a simple Brownian motion and for simplicity we will only consider the process for the natural logarithm of the productivities. We let the drift be defined as μ and the standard deviation as σ. The probability density for ideas with a productivity ϕ, that start from a point $\bar{z}$ and follow a simple Brownian motion is simply

$$\begin{aligned} p\left(\phi, a|\bar{z}\right) &= \frac{1}{\sigma_z\sqrt{\alpha 2\pi}}\exp\left\{-\frac{\left(\frac{\phi-\bar{z}-\mu a}{\sigma\sqrt{\alpha}}\right)^2}{2}\right\} = \\ & \frac{1}{\sigma\sqrt{\alpha 2\pi}}\exp\left\{-\frac{\left(\phi-\bar{z}-\mu a\right)^2}{\sigma_z^2 2a}\right\}\ . \end{aligned}$$

with support $(-\infty, +\infty)$.

The entry rate is $g_\eta\left(1-\alpha\right).$ Therefore, in order to find the cross-sectional distribution for ϕ's we have to integrate the density considering a population weight for differenent generations of firms. Namely we need a weight $\exp\left\{-g_\eta\left(1-\alpha\right)a\right\}$ for firms of age a. This gives us:

$$g\left(\phi|\bar{z}\right) = \int_0^{+\infty}\frac{e^{-g_\eta(1-\alpha)a}}{\sigma_z\sqrt{a2\pi}}\exp\left\{-\frac{\left(\phi-\bar{z}-\mu a\right)^2}{\sigma_z^2 2a}\right\}da =$$

$$\int_0^{+\infty}\frac{e^{-g_\eta(1-\alpha)a}}{\sigma_z\sqrt{a2\pi}}\exp\left\{-\frac{\left(\phi-\bar{z}\right)^2-2\left(\phi-\bar{z}\right)\mu a+\left(\mu a\right)^2}{\sigma_z^2 2a}\right\}da =$$

$$\frac{1}{\sigma_z}\exp\left\{\frac{\left(\phi-\bar{z}\right)\mu}{\sigma_z^2}\right\}\int_0^{+\infty}\frac{1}{\sqrt{a2\pi}}\exp\left\{-\left(\frac{\left(\phi-\bar{z}\right)^2}{\sigma_z^2 2a}+\frac{\left(\mu^2+\sigma_z^2 2g_\eta\left(1-\alpha\right)\right)\left(a\right)^2}{\sigma_z^2 2a}\right)\right\}da =$$

$$\frac{1}{\sigma_z}\exp\left\{\frac{\left(\phi-\bar{z}\right)\mu}{\sigma_z^2}\right\}\int_0^{+\infty}\frac{1}{\sqrt{a2\pi}}\exp\left\{-\left(\frac{\sqrt{\left(\phi-\bar{z}\right)^2}}{\sigma_z\sqrt{2a}}-\sqrt{\frac{\left(\mu^2+\sigma_z^2 2g_\eta\left(1-\alpha\right)\right)a}{\sigma_z^2 2}}\right)^2-\frac{\sqrt{\left(\phi-\bar{z}\right)^2}}{\sigma_z}\frac{\sqrt{\left(\mu^2+\sigma_z^2 2g_\eta\left(1-\alpha\right)\right)}}{\sigma_z}\right\}da$$

$$\frac{1}{\sigma_z}\exp\left\{\frac{\left(\phi-\bar{z}\right)\mu}{\sigma_z^2}-\frac{\sqrt{\left(\phi-\bar{z}\right)^2}}{\sigma_z}\frac{\sqrt{\left(\mu^2+\sigma_z^2 2g_\eta\left(1-\alpha\right)\right)}}{\sigma_z}\right\}\int_0^{+\infty}\frac{1}{\sqrt{a2\pi}}\exp\left\{-\left(\frac{\sqrt{\left(\phi-\bar{z}\right)^2}}{\sigma_z\sqrt{2a}}-\sqrt{\frac{\left(\mu^2+\sigma_z^2 2g_\eta\left(1-\alpha\right)\right)a}{\sigma_z^2 2}}\right)^2\right\}da$$

Now notice that if we set $x=\sqrt{a}$, $dx=\frac{1}{2}\frac{1}{\sqrt{a}}da$ we have

$$\begin{aligned} \int_0^{+\infty}\frac{1}{\sqrt{a2\pi}}\exp\left\{-\frac{\left(\frac{\sqrt{(\phi-\bar{z})^2}}{\sqrt{a}}-\sqrt{a}\sqrt{\left(\mu^2+\sigma_z^2 2g_\eta\left(1-\alpha\right)\right)}\right)^2}{\sigma_z^2 2}\right\}da &= \\ \int_0^{+\infty}\frac{2}{\sqrt{2\pi}}\exp\left\{-\frac{\left(\frac{\sqrt{(\phi-\bar{z})^2}}{x}-x\sqrt{\mu^2+\sigma_z^2 2g_\eta\left(1-\alpha\right)}\right)^2}{\sigma_z^2 2}\right\}dx &= \end{aligned}$$

and thus

$$g\left(\phi|\bar{z}\right)=\frac{1}{\sigma_z}\exp\left\{\frac{\left(\phi-\bar{z}\right)\mu}{\sigma_z^2}\right\}\int_0^{+\infty}\frac{1}{\sqrt{a2\pi}}\exp\left\{-\left(\frac{\left(\phi-\bar{z}\right)^2}{\sigma_z^22a}+\frac{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)\left(a\right)^2}{\sigma_z^22a}\right)\right\}da=$$

$$\exp\left\{\frac{\left(\phi-\bar{z}\right)\mu}{\sigma_z^2}\right\}\frac{2}{\sigma_z\sqrt{2\pi}}\int_0^{+\infty}\exp\left\{-\frac{\left(\phi-\bar{z}\right)^2}{\sigma_z^22x^2}-\frac{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)x^2}{\sigma_z^22}\right\}dx$$

Now we have the following result for the last integral

$$\int_0^{+\infty}e^{-a/x^2-bx^2}dx=\frac{1}{4\sqrt{b}}\left(\begin{array}{c}\sqrt{\pi}\left[e^{-2\sqrt{a}\sqrt{b}}\left(\operatorname{erf}\left(\sqrt{b}x-\frac{\sqrt{a}}{x}\right)+1\right)\right]+\\ e^{2\sqrt{a}\sqrt{b}}\left(\operatorname{erf}\left(\sqrt{b}x+\frac{\sqrt{a}}{x}\right)-1\right)\end{array}\Bigg|_0^{+\infty}\right)$$

where $\operatorname{erf}\left(x\right)$ is the error function, $\operatorname{erf}\left(x\right)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ (and $\operatorname{erf}\left(+\infty\right)=1$, $\operatorname{erf}\left(-\infty\right)=-1$). This means that

$$\begin{aligned}\int_0^{+\infty}e^{-a/x^2-bx^2}dx &= \frac{1}{4\sqrt{b}}\left(\begin{array}{c}\sqrt{\pi}e^{-2\sqrt{a}\sqrt{b}}2+0\\ \\ 0+0\end{array}\right)=\\ & \quad\frac{1}{2\sqrt{b}}\left(\sqrt{\pi}e^{-2\sqrt{a}\sqrt{b}}\right)\end{aligned}$$

Given the above $g\left(\phi|\bar{z}\right)$ becomes

$$g\left(\phi|\bar{z}\right)=\exp\left\{\frac{\left(\phi-\bar{z}\right)\mu}{\sigma_z^2}\right\}\frac{2}{\sigma_z\sqrt{2\pi}}\int_0^{+\infty}\exp\left\{-\frac{\left(\phi-\bar{z}\right)^2}{\sigma_z^22x^2}-\frac{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)x^2}{\sigma_z^22}\right\}dx=$$

$$\exp\left\{\frac{\left(\phi-\bar{z}\right)\mu}{\sigma_z^2}\right\}\frac{2}{\sigma_z\sqrt{2\pi}}\frac{1}{2\sqrt{\frac{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)}{\sigma_z^22}}}\left(\sqrt{\pi}e^{-2\sqrt{\frac{\left(\phi-\bar{z}\right)^2}{\sigma_z^22}}\sqrt{\frac{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)}{\sigma_z^22}}}\right)=$$

$$\frac{1}{\sqrt{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)}}e^{\left(\phi-\bar{z}\right)\frac{\mu}{\sigma_z^2}-\left|\phi-\bar{z}\right|\frac{1}{\sigma_z^2}\sqrt{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)}}=$$

$$\frac{1}{\sqrt{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)}}e^{\left(\phi-\bar{z}\right)\frac{\mu}{\sigma_z^2}-\left|\phi-\bar{z}\right|\frac{1}{\sigma_z^2}\sqrt{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)}}=$$

$$\left\{\begin{array}{l}\frac{e^{\left(\phi-\bar{z}\right)\left(\frac{\mu}{\sigma_z^2}-\frac{1}{\sigma_z^2}\sqrt{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)}\right)}}{\sqrt{\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)}}\text{ if }\phi\geq\bar{z}\\ \frac{e^{\left(\phi-\bar{z}\right)\left(\frac{\mu}{\sigma_z^2}+\frac{1}{\sigma_z^2}\sqrt{\left(\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)\right)}\right)}}{\sqrt{\mu^2+\sigma_z^22g_\eta\left(1-\alpha\right)}}\text{ if }\phi<\bar{z}\end{array}\right.$$

Thus, the density is

$$g(\phi|\bar{z}) = \frac{\min\left\{e^{\theta_1(\phi-\bar{z})}, e^{-\theta_2(\phi-\bar{z})}\right\}}{\sqrt{\mu^2 + \sigma_z^2 2 g_\eta (1-\alpha)}},$$

$$\theta_{1,2} = \pm\frac{\mu}{\sigma_z^2} + \sqrt{\frac{\mu^2}{\sigma_z^4} + 2\frac{g_\eta(1-\alpha)}{\sigma_z^2}}.$$

We can also express it as a probability density

$$p(\phi|\bar{z}) = \frac{\theta_1\theta_2}{\theta_1+\theta_2}\min\left\{e^{\theta_1(\phi-\bar{z})},\ e^{-\theta_2(\phi-\bar{z})}\right\},$$

where

$$\frac{\theta_1\theta_2}{\theta_1+\theta_2} = \frac{2\frac{g_\eta(1-\alpha)}{\sigma_z^2}}{\sqrt{\frac{\mu^2}{\sigma_z^4} + 2\frac{g_\eta(1-\alpha)}{\sigma_z^2}}} = \frac{2g_\eta(1-\alpha)}{\sqrt{\mu^2 + 2\sigma_z^2 g_\eta(1-\alpha)}}\ .$$

Notice that in order for the distribution function to be properly defined a restriction in θ_1, θ_2 has to be imposed and in particular that $\theta_1, \theta_2 > 0$. This implies that $g_\eta(1-\alpha) > 0$.

Finally, if we assume that the initial $\bar{z}$ is drawn randomly from a distribution $G(\bar{z})$ we can also derive

$$p(\phi) = \int p(\phi|\bar{z})\,G(\bar{z})\,d\bar{z}\ .$$

This is done in (Reed 2002).

References

Arkolakis, C. (2007): "Market Penetration Costs and Trade Dynamics," *mimeo, Yale University.*

Reed, W. J. (2001): "The Pareto, Zipf and Other Power Laws," *Economics Letters*, 74(1), 15–19.

——— (2002): "On the Rank-Size Distribution for Human Settlements," *Journal of Regional Science*, 42(1), 1–17.