Differentiating an Integral

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Leibniz Rule. If $\Phi(t) = \int_{a(t)}^{b(t)} f(z,t)dz$, then

$$\frac{d\Phi}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(z,t)}{\partial t} dz + f\left(b(t),t\right)\frac{db}{dt} - f\left(a(t),t\right)\frac{da}{dt}.$$

That's the rule; now let's see why this is so. We start by reviewing some things you already know. Let f be a smooth univariate function (i.e., a "nice" function of one variable). We know from the fundamental theorem of calculus that if the function ϕ is defined as

$$\phi\left(x\right) = \int_{c}^{x} f(z)\ dz$$

(where c is a constant) then

$$\frac{d\phi\left(x\right)}{dx} = f(x). \tag{1}$$

You can gain some really useful intuition for this by remembering that the integral $\int_c^x f(z)\ dz$ is the area under the curve $y = f(z)$, between $z = a$ and $z = x$. Draw this picture to see that as x increases infinitessimally, the added contribution to the area is $f\left(x\right)$.

Likewise, if

$$\phi\left(x\right) = \int_{x}^{c} f(z)\ dz$$

then

$$\frac{d\phi\left(x\right)}{dx} = -f(x). \tag{2}$$

Here, an infinitessimal increase in x reduces the area under the curve by an amount equal to the height of the curve.

Now suppose f is a function of two variables and define

$$\phi(t) = \int_{a}^{b} f(z,t)dz \tag{3}$$

where a and b are constants. Then

$$\frac{d\phi\left(t\right)}{dt} = \int_{a}^{b} \frac{\partial f(z,t)}{\partial t} dz \tag{4}$$

i.e., we can swap the order of the operations of integration and differentiation and "differentiate under the integral." You've probably done this "swapping" before. And here we are being very explicit about the fact that we are differentiating only with respect to the second argument of f.

Now let's put all this together. Take the function ϕ in (3) above and note that this function also depends on the values of a and b, not just t. So let's write it in a way that explicitly accounts for this:

$$\phi\left(t, a, b\right). \tag{5}$$

Now suppose the limits of integration a and b themselves depend on t, so that $a = a\left(t\right)$, $b = b\left(t\right)$. Let

$$\Phi(t) = \phi\left(t, a(t), b(t)\right) = \int_{a(t)}^{b(t)} f(z,t)dz. \tag{6}$$

Our goal now is to obtain the derivative of $\Phi(t)$ with respect to t. (Why is this our goal? Because this is exactly the kind of derivative that shows up in the Leiniz rule).

Using the definition (6), by the chain rule we obtain

$$\frac{d\Phi(t)}{dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial a}\frac{da}{dt} + \frac{\partial\phi}{\partial b}\frac{db}{dt}.$$

Using the results (1), (2), and (4), this gives

$$\frac{d\Phi(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(z,t)}{\partial t} dz + f\left(b(t), t\right)\frac{db}{dt} - f\left(a(t), t\right)\frac{da}{dt}$$

which is the Leibniz rule.

Drawing a picture (again remembering that an integral represents the area under a curve) will be helpful here. There are three effects of a marginal change in t on the area defined by the integral in (6): the integrand changes over the whole region of integration (the first term above), the right boundary is moved slightly (the second term), and the left boundary is moved slightly (the final term).

Following is a simple example that is typical of the limited use of the Leibniz rule that we'll need for this class:

Example 1 *Suppose the market inverse demand curve is given by the smooth function $P\left(q\right)$, each firm's cost function is $cq + f$, and equilibrium quantity can be represented as a function of n (the number of firms) with a smooth function $q\left(n\right)$. Then total surplus (net of fixed cost) is given by*

$$S(n) = \int_0^{q(n)} P\left(Q\right)\ dQ - cq(n) - nf.$$

To see the effect on total surplus of a change in n, we need to differentiate $S\left(n\right)$ with respect to n. By the Leibniz rule,

$$\frac{dS(n)}{dq} = \frac{dq(n)}{dn}\left[P(q(n)) - c\right] - f.$$