Notes on the Coase Conjecture[1]

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Following is a sketch of a proof of the Coase conjecture. This is a difficult result to prove, and even this partial sketch is not simple.

Suppose a durable good monopolist with fixed supply and zero marginal cost maximizes profit by setting a sequence of nonincreasing prices $p_1, p_2, \ldots$ in periods $1, 2, \ldots$. Given these prices, let Q_t denote the *cumulative* quantity sold at the end of period t, so that $Q_t - Q_{t-1}$ is the quantity sold in period t. Time itself is continuous but opportunities for trade occur at discrete points in time, beginning at time 0. We let Δ denote the time between trading periods. Consumers and the monopolist have a per-period discount factor

$$\delta = \exp(-r\Delta)$$

where $r > 0$. Note that $\delta \to 1$ as $\Delta \to 0$; i.e., discounting between periods vanishes as the length of time between periods does. (See the separate notes on continous time discounting). Let π_t represent the present value (from the perspective of period t) of the monopolist's profit from period t forward.

Assume that each consumer uses a simple cutoff rule: consumer j buys in period t if and only if $p_t \leq c_j$ for some c_j. Since the cutoff price c_j is chosen optimally by consumer j, consumers with higher valuations for the good will have higher cutoff prices. Let v_t denote the valuation of the good for a consumer who is *indifferent* between buying in periods $t-1$ and t, given the monopolist's price sequence. Hence, it is consumers with valuations between v_t and v_{t+1} who purchase in period t.

Lemma. Fix any time $\epsilon > 0$. I first claim that as Δ becomes sufficiently small it must be the case that for any $\eta > 0$, we can find a period t such that (a) period $t+2$ takes place before time ϵ and (b)

$$Q_{t+1} - Q_{t-1} < \eta. \tag{1}$$

This lemma says that by letting the number of time periods before time ϵ approach infinity, we can ensure that there are two consecutive periods before time ϵ in which the quantity sold adds up to η or less. Why is the lemma true? The proof is by contradiction:

Proof. If the claim were false, there would be arbitarily many trading periods prior to time ϵ in which a quantity of at least η is sold. This would mean that the quantity sold prior to time ϵ would be going to infinity as $\Delta \to 0$, which cannot happen with fixed supply.

[1]This is based on the presentation in chapter 1 of Jean Tirole's *Industrial Organization*, MIT Press 1988.

Now, for the monopolist's pricing strategy to be optimal, it must be the case that it never wants to "speed up" the process of moving down the demand curve, for example by jumping to price p_{t+1} in period $t-1$ and then continuing through the original price sequence, charging prices $p_{\tau+1}$ in periods $\tau = t, t+1, \ldots$.[2] Writing this requirement down, for all t we must have

$$p_t(Q_t - Q_{t-1}) + \delta p_{t+1}(Q_{t+1} - Q_t) + \delta^2 \pi_{t+2} \geq p_{t+1}(Q_{t+1} - Q_{t-1}) + \delta \pi_{t+2}.$$

Rearranging this gives

$$-\left(p_t - p_{t+1}\right) Q_{t-1} + \left(p_t - \delta p_{t+1}\right) Q_t - (1-\delta) p_{t+1} Q_{t+1} \geq \delta(1-\delta)\pi_{t+2.} \tag{2}$$

Now note that the indifference condition that defines v_{t+1} is

$$v_{t+1} - p_t = \delta(v_{t+1} - p_{t+1}).$$

Rearranging this gives

$$p_t - \delta p_{t+1} = (1-\delta) v_{t+1} \tag{3}$$

and also (subtracting p_{t+1} from both sides)

$$p_t - p_{t+1} = (1-\delta)(v_{t+1} - p_{t+1}). \tag{4}$$

Substituting (3) and (4) into (2) gives (after canceling the term $(1-\delta)$ and collecting terms)

$$v_{t+1}(Q_t - Q_{t-1}) - p_{t+1}(Q_{t+1} - Q_{t-1}) \geq \delta \pi_{t+2}.$$

Since $Q_{t+1} \geq Q_t$, this implies that for *every* t

$$v_{t+1}(Q_{t+1} - Q_{t-1}) - p_{t+1}(Q_{t+1} - Q_{t-1}) \geq \delta \pi_{t+2}.$$

By the Lemma, this implies that for any ϵ and arbitrarily small η there is *some* t such that $(t+2)\,\Delta < \epsilon$ and

$$(v_{t+1} - p_{t+1})\,\eta \geq \delta \pi_{t+2}.$$

The left side can be arbitrarily close to zero since η can. Since $\delta \to 1$ as $\Delta \to 0$, this requires that the total profits earned after time ϵ be arbitrarily small. This is possible only if the quantity sold after time ϵ at any price strictly above marginal cost is arbitrarily small. Finally, since this

[2] This is, of course, just one of many inequalities implied by the fact that the chosen price sequence is optimal: theere can be no profitable deviation from this strategy. Note that when this particular deviation is taken, π_{t+2} becomes the present value of profits obtained from period $t+1$ forward.

must be true even when ϵ itself is arbitrarily close to zero, the monopolist's market power must be vanishing "in the twinkling of an eye." If you made it this far, you should feel good!

Comment: The intuition often given for the Coase conjecture is based on consumers' becoming "patient" as the time between periods vanishes: consumers are always willing to postpone buying to get a lower price in the future, so the monopolist might as well just set price=marginal cost immediately. However, this intuition is incomplete because the seller also becomes patient in the same sense consumers do. The result depends on combining optimal behavior by both consumers and the firm. An additional key to the argument above is the presence of a continuum of consumer valuations; i.e., that inducing the marginal consumer to buy now instead of next period can always be achieved with an arbitrarily small reduction in the current-period price. If instead consumer valuations were concentrated on a discrete set of values, the monopolist would have to lower price by a discrete amount to attract any additional sales. In that case the Coase conjecture can be overturned. An intuition is that the monopolist must lower price by a discrete amount to make additional sales in period t, but gains little from this acceleration of sales if Δ is small. Church and Ware's textbook talks some about this.