Notes on Continuous Time Discounting

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You have seen examples of problems in **discrete time** before, where we think of distinct time periods $p = 1, 2, \ldots$. There, we are used to thinking about the relation between *value* (of an asset, or of consumption at some specified date) and time using a per-period discount rate (or interest rate) i. Let $v(p)$ denote the value in period p. Then the following familiar relationships are equivalent:

$$v(p)(1+i) = v(p+1) \tag{1}$$

$$v(p) = \frac{1}{(1+i)} v(p+1) \tag{2}$$

$$i = \frac{v(p+1) - v(p)}{v(p)}. \tag{3}$$

Note that in (3), the interest rate i is equal to the percentage change in value per period.

An alternative way to think about time is as a continuous variable t. This may seem more "realistic," and in some cases a **continuous time** analysis is also more useful in economic models. The notion of an interest rate or discount rate in continuous time is very similar. With continous time, there are no arbitrary time periods and therefore no natural notion of a "per-period" discount rate. However, we will often still want to compare values at different points in time. To do this we will introduce the notion of an "instantaneous" discount rate (or, instantaneous interest rate) r.

Let's start from what we know in the discrete time case. Image there that the discrete period period p actually begins at (continous) time t. Now think of rewriting equation (3) by writing r on the left; on the right, replace the per period rate of change in value $v(p+1) - v(p)$ with the instantaneous rate of change in value, $\frac{d}{dt}v(t)$. This gives us

$$r = \frac{\frac{d}{dt}v(t)}{v(t)}. \tag{3'}$$

Like before, this describes the rate of change in value, expressed in percentage terms.

Now rewrite (3') as

$$r = \frac{d}{dt} \ln (v(t)). \tag{(3'')}$$

To use this definition to compare values at times Δ apart, observe that (integrating both sides of

(3”))

$$
\begin{aligned}
\int_{s=t}^{t+\Delta} r\ ds &= \int_{s=t}^{t+\Delta} \frac{d}{dt} \ln\left(v(s)\right)\ ds \\
r\Delta &= \ln\left(v(t+\Delta)\right) - \ln\left(v(t)\right) \\
&= \ln\left(\frac{v(t+\Delta)}{v(t)}\right)
\end{aligned}
$$

so that

$$e^{r\Delta} = \frac{v(t+\Delta)}{v(t)}.$$

Now we have something, because this is equivalent to

$$v(t)e^{r\Delta} = v(t+\Delta) \tag{1'}$$

or

$$v(t) = v(t+\Delta)e^{-r\Delta}. \tag{2'}$$

So now we see how too make the usual present and future value calculations in continuous time. In continuous time $e^{-r\Delta}$ plays the same role that the discount factor $\frac{1}{1+i}$ does in discrete time. More precisely, if the periods in the discrete time model are Δ apart in continuous time,

$$e^{-r\Delta} = \frac{1}{1+i}.$$

Finally, we can use this to think precisely about what happens as the length of time “between periods” changes (i.e., THE POINT of all this!) In class, I claimed that it was natural to think about i getting very close to zero as the length of time between periods shrinks to zero. You can see this now:

$$\lim_{\Delta \to 0} e^{-r\Delta} = 1$$

i.e., as the length of time between periods goes to zero, the discounting of future payments also vanishes.