Leibniz Rule for differentiating an Integral

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Leibniz Rule. If $\Phi(t) = \int_{a(t)}^{b(t)} f(z,t)dz$, then

$$\frac{d\Phi}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(z,t)}{\partial t} dz + f\left(b(t), t\right) \frac{db}{dt} - f\left(a(t), t\right) \frac{da}{dt}.$$

Note that the rule uses partial differentiation, i.e., differentiation with respect to one argument of a function holding the other arguments fixed. If this puzzles you, may want to look at the posted notes on partial differentiation.

That's the rule; now let's see why this is so. Let f be a smooth univariate function (i.e., a "nice" function of one variable). We know from the fundamental theorem of calculus that if the function ϕ is defined as

$$\phi\left(x\right) = \int_c^x f(z)\ dz$$

(where c is a constant) then

$$\frac{d\phi\left(x\right)}{dx} = f(x). \tag{1}$$

Likewise, if

$$\phi\left(x\right) = \int_x^c f(z)\ dz$$

then

$$\frac{d\phi\left(x\right)}{dx} = -f(x). \tag{2}$$

Now suppose f is a function of two variables and define

$$\phi(t) = \int_a^b f(z,t)dz$$

where a and b are constants. Then

$$\frac{d\phi\left(t\right)}{dt} = \int_a^b \frac{\partial f(z,t)}{\partial t} dz \tag{3}$$

i.e., we can swap the order of the operations of integration and differentiation and "differentiate under the integral."

Now let's put all this together. Take the final function ϕ above and note that this function also depends on the values of a and b. So let's write it in a way that explicitly accounts for this:

$$\phi\left(t, a, b\right). \tag{4}$$

Now suppose the limits of integration a and b themselves depend on t :

$$\Phi(t) = \phi\left(t, a(t), b(t)\right) = \int_{a(t)}^{b(t)} f(z,t)dz. \tag{5}$$

Our goal is to obtain the derivative of $\Phi(t)$ with respect to t. Using the definition (5) chain rule, we obtain

$$\frac{d\Phi(t)}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial a}\frac{da}{dt} + \frac{\partial \phi}{\partial b}\frac{db}{dt}.$$

Using the results (1), (2), and (3), this gives

$$\frac{d\phi}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(z,t)}{\partial t} dz + f\left(b(t), t\right)\frac{db}{dt} - f\left(a(t), t\right)\frac{da}{dt}$$

which is the Leibniz rule.

Intuitively, there are three effects of a marginal change in t on the area defined by the integral in (5): the integrand changes over the whole region of integration (the first term), the right boundary is moved slightly (the second term), and the left boundary is moved slightly (the final term). Drawing a picture (and remembering that an integral represents the area under a curve) will be helpful here.

Following is a simple example that is typical of the limited use of this result we'll need for this class:

Example 1 *Suppose the market inverse demand curve is given by $P(q)$, each firm's cost function is $cq + f$, and equilibrium quantity can be represented as a function of n (the number of firms) with $q(n)$. Then total surplus is given by*

$$S(n) = \int_{0}^{q(n)} P(Q)\ dQ - cq(n) - nf.$$

To see the effect on total surplus of a change in n, we need to differentiate $S(n)$ with respect to n. By the Leibniz rule,

$$\frac{dS(n)}{dq} = \frac{dq(n)}{dn}\left[P(q(n)) - c\right] - f.$$