Answer Key: Problem Set 1

January 29, 2008

Problem 1

(a)

Demand:

$$P = \alpha - \beta q$$

Revenue:

$$Pq = (\alpha - \beta q)q$$

Marginal Cost: $MC = c$ and there is no fixed cost

The monopolist maximizes:

$$\pi = Pq - cq = (\alpha - \beta q)q - cq$$

The first order condition with respect to q is

$$\alpha - 2\beta q - c = 0$$

Solving for q:

$$q^* = \frac{(\alpha - c)}{2\beta}$$

Substituting this into demand function yields:

$$P^* = \frac{\alpha + c}{2}$$

b) Remember the first order condition (FOC) of the profit maximization is:

$$\frac{dp}{dq}q + (p(q) - c) = 0$$

or

$$\frac{dp}{dq}\frac{q}{p} + 1 = \frac{c}{p}$$

so that

$$1 - \varepsilon = \frac{c}{p}$$

$$\varepsilon = -\frac{dp}{dq}\frac{q}{p}$$

so that solving for price we get that

$$p = \frac{c}{1 - \varepsilon}$$

In this case (linear demand),

$$\varepsilon = -\frac{1}{\beta}\frac{p^*}{q^*}$$

c) Profits:

$$\begin{aligned} \pi^* &= P^* q^* - c q^* \\ &= \frac{(\alpha - c)(\alpha + c)}{4\beta} - c\frac{\alpha - c}{2\beta} \\ &= \frac{(\alpha - c)^2}{4\beta} \end{aligned}$$

d) Consumer Surplus (see Figure at end of text):

$$\begin{aligned} CS &= (\alpha - \frac{\alpha + c}{2})\frac{\alpha - c}{2\beta}\frac{1}{2} \\ &= \frac{\alpha - c}{2}\frac{\alpha - c}{2\beta}\frac{1}{2} \\ &= \frac{(\alpha - c)^2}{8\beta} \end{aligned}$$

e)Deadweight loss:

$$
\begin{aligned}
DWL &= \frac{(q^e - q^*)(P^* - c)}{2} \\
&= (\frac{\alpha - c}{\beta} - \frac{\alpha - c}{2\beta})(\frac{\alpha + c}{2} - c)\frac{1}{2} \\
&= \frac{(\alpha - c)^2}{8\beta}
\end{aligned}
$$

f) $P = 10 - q$, hence $q = 10 - P$. Market Demand is calculated by summing over all individual demands:

$$
\begin{aligned}
Q &= 100 * (10 - P) \\
P &= 10 - Q/100
\end{aligned}
$$

We have $\alpha = 10$ and $\beta = \frac{1}{100}$. Using your results from part (b) and (d) you get

$$
\begin{aligned}
\pi^* &= 25(10 - c)^2 \\
DWL &= 25(10 - c)^2
\end{aligned}
$$

g) The monopolist would set price equal to marginal cost c and set a fixed fee equal to the consumer surplus $\frac{q^e(\alpha - c)}{2}$ that would result were there no fixed fee. The deadweight loss equals zero. Profits then equal the fixed fee since

$$\pi^* = A + cq - cq$$

where A is the fixed fee. We get:

$$
\begin{aligned}
\pi^* &= \frac{(\alpha - c)^2}{2\beta} \\
&= 50(10 - c)^2
\end{aligned}
$$

Problem 2

Let's setup the problem. The monopolist sets a two part tariff $T = a + bq$. In fact, he chooses a and b to maximize profits:

$$\max_{a,b} \{a + bq_1 + a + bq_2 - c(q_1 + q_2)\}$$

or

$$\max_{a,b} \{2a + (b - c)(q_1 + q_2)\}$$

where q_i is the demand of consumer type i, $i = 1, 2$. We are going to use to kinds of conditions to solve this problem. The first has to do with the fact that q_1 and q_2 are chosen optimally from the consumers and depend on the monopolist's actions (a and b). So, we derive consumer i's demand. Each consumer maximizes

$$u_i(q, T) = \theta_i v(q) - (a + b)q$$

The first order condition yields

$$(1 - q)\theta_i - b = 0$$

and hence $q_i^*(b) = 1 - \frac{b}{\theta_i}$.

We also have some information on how the monopolist sets a: he wants to extract all consumer surplus from the type 1 consumer. Indeed, to make sure consumers buy, the monopolist must make sure their utility is non-negative, or:

$$\theta_i v(q_i) - bq_i - a \geq 0, \; i = 1, 2$$

or

$$a \leq \theta_i v(q_i) - bq_i, \; i = 1, 2$$

If this inequality is satisfied for the low type, it will also be satisfied for the high type (another way to say this: setting a from the high type's inequality makes sure the low type won't buy).

The monopolist, however, has no reason to leave positive surplus to the consumer: high a is better than low a and so:

$$\begin{aligned} a(b) &= \theta_1 v(q_1^*) - bq_1^*(b) \\ &= \theta_1 \frac{1 - [1 - (1 - \frac{b}{\theta_1})]^2}{2} - b(1 - \frac{b}{\theta_1}) \\ &= \frac{(\theta_1 - b)^2}{2\theta_1} \end{aligned}$$

She then maximizes aggregate profits :

$$\begin{aligned}\pi &= 2a(b) + (b-c)\left(q_1^*(b) + q_2^*(b)\right) \\ &= 2\frac{(\theta_1 - b)^2}{2\theta_1} + (b-c)[(1-\frac{b}{\theta_1}) + (1-\frac{b}{\theta_2})]\end{aligned}$$

The first order condition with respect to b yields:

$$-2\frac{(\theta_1 - b)}{\theta_1} + [(1-\frac{b}{\theta_1}) + (1-\frac{b}{\theta_2})] - \frac{(b-c)}{\theta_1} - \frac{(b-c)}{\theta_2} = 0$$

Solving for b yields:

$$b^* = \frac{c}{2}\frac{\theta_1 + \theta 2}{\theta_1}$$

The optimal two part tariff then becomes

$$\begin{aligned}T^* &= a(b^*) + b^* q \\ &= \frac{(2\theta_1^2 - c[\theta_1 + \theta_2])^2}{8\theta_1^3} + \frac{c}{2}\frac{\theta_1 + \theta_2}{\theta_1} q\end{aligned}$$

Problem 3

a) Total demand is $Q = 28 - 7P$, i.e.: $P = 4 - \frac{Q}{7}$ Use your results from Problem 1(a) to get $Q = 7$ and $P = 3$.

b) Again use your results from Problem 1(a). For students we have $p = 3.6 - \frac{q}{5}$. So $Q_S = 4$ and $P_S = 2.8$. For non-students we have $p = 5 - \frac{q}{2}$and $Q_{NS} = 3$ and $P_{NS} = 3.5$

c) The Two-part tariff will extract all surplus with the fixed cover charge equalling the surplus and a price per drink equal to marginal cost. Students: $Q(2) = 8$, $CS(2) = \frac{(3.6-2)8}{2} = 6.40$ (note: students won't actually get this surplus; this is just the surplus they would get if there were no cover charge.) Non-students: $(2) = 6$, $CS(2) = \frac{(5-2)6}{2} = 9$ (same comment applies).

d) He will charge an \$6.4 cover to extract all the rent from the students.

e) Before midnight: Average demand is:

$$\begin{aligned} Q &= \frac{(18-5P)2}{7} + \frac{(10-2P)5}{7} \\ &= \frac{86}{7} - \frac{20P}{7} \end{aligned}$$

and thus $P = 4.3 - 7Q/20$. Thus we have $P = 3.15$, and profit $\Pi = 3.78$.

After midnight: Since those who remain after midnight are all students (there is only one type consumer then), p = 2.80 (just as what we have seen in part (b)), Q = 4, and profit $\Pi = 4 * (2.8 - 2) = 3.2$. Profits in part (d) are 6.40. In part (c) they are 7.7.

Note: to get aggregate demand all you have to do is add up the quantities demanded from the several different consumer types (if such exist). Here we assume that all consumers of the same type are identical and so they will all choose the same quantity q_i. So in this case if you write up $q_1 + q_2$, you imply that the types are in equal proportions. If they are not, you have to weigh the individual quantities of each consumer of each type by the proportions of the types $(2/7, 5/7)$.

Problem 4

a) In this problem, we have two types of consumers: Low demand consumers (share λ): $Q_L = 10 - 2P_L$. High demand consumers (share $1 - \lambda$): $Q_H = 10 - P_H$ The consumer surplus for price P is given by $CS_L(P) = \frac{1}{2\beta}(a - P_L)^2 = (5 - P_L)^2$. The monopolist maximizes her profits (setting the fixed fee equal to CS_L):

$$\underset{P}{\text{MAX}}\, CS_L(P) + [\lambda Q_L(P) + (1-\lambda)Q_H(P)](P - c)$$

i.e.:

$$\underset{P}{\text{MAX}}(5 - P)^2 + [10 - P(1+\lambda)](P - 2)$$

F.O.C:

$$-2(5 - P) + 10 - P - \lambda P + (P - 2)(-1 - \lambda) = 0$$

solving for P yields

$$P = (1+\lambda)/\lambda$$

Note that $\frac{1+\lambda}{\lambda} > 2 = c$. The fixed fee equals $CSL(P) = [5 - \frac{1+\lambda}{\lambda}]^2$. We can calculate monopolist's profit as:

$$\pi = [5 - \frac{1+\lambda}{\lambda}]^2 + [10 - \frac{(1+\lambda)^2}{\lambda}]\frac{1-\lambda}{\lambda}$$

(b) When only high-demand consumers purchase the good, the charged fixed fee

would be equal to $CS_H(2) = \frac{(10-2)^2}{2} = 32$. The profit will be $32(1-\lambda)$.

(d) From result in part (a), we can calculate P. When $\lambda = \frac{1}{2}$:

$$\begin{aligned}
\pi &= [5 - (1+0.5)2]^2 + [10 - 2(1+.5)^2](1-.5)2 \\
&= 9.5 \\
&< 16 \\
&= 32(1-.5)
\end{aligned}$$

Hence, monopolist will choose (b), i.e. $P = c$.

When $\lambda = \frac{3}{4}$,

$$\begin{aligned}
\pi &= [5 - \frac{4(1+.75)}{3}]^2 + [10 - \frac{4(1+.75)^2}{3}]\frac{(1-.75)4}{3} \\
&= 9.083 \\
&> 8 \\
&= 32(1-.75)
\end{aligned}$$

Hence, the monopolist will choose (a), i.e. sell to both groups of consumers.

Problem 5

a) see graph

b)Let us lower q_1 slightly to q_1' Since q_1 was at tangency between the iso-profit curve and the indifference curve, for a small change, the profit on type 1 consumer will not change. Now hold q_2 fixed but raise T_2 to make type 2 indifferent between (q_2, T_2') and (q_1', T_1') : Note that IR is binding for type 1 :

$$\begin{aligned} T_1 &= \theta_1 v(q_1) \\ T_1' &= \theta_1 v(q_1') \end{aligned}$$

And type 2's IC constraint is also binding

$$\begin{aligned} \theta_2 v(q_2) - T_2 &= \theta_2 v(q_1) - T_1 \\ \theta_2 v(q_2) - T_2' &= \theta_2 v(q_1') - T_1' \end{aligned}$$

So that

$$\begin{aligned} T_2 &= \theta_2(v(q_2) - v(q_1)) + T_1 \\ T_2' &= \theta_2(v(q_2) - v(q_1')) + T_1' \end{aligned}$$

And

$$\begin{aligned} \pi_2 &= T_2 - cq_2 = \theta_2(v(q_2) - v(q_1)) + T_1 - cq_2 \\ \pi_2' &= T_2' - cq_2 = \theta_2(v(q_2) - v(q_1')) + T_1' - cq_2 \\ \pi_2' - \pi_2 &= (\theta_2 - \theta_1)(v(q_1) - v(q_1')) > 0 \end{aligned}$$

So lowering $q_1$and leaving q_2 unchanged increases the monopolist's profits on the type 2 consumer without changing that on the type 1 consumer, so that it increases the monopolist's total profits.

Another approach is to show that the monopoly first order condition (in terms of profit) evaluated at q_1 is not zero. First, the monopoly chooses (q_1, T_1) to max profit from consumer 1:

$$\begin{gathered} \max \Pi_1 = T_1 - c * q_1 \\ s.t. \quad \theta_1 v(q_1) - T \geq 0 \end{gathered}$$

This implies that $T_1 = \theta_1 v(q_1)$ and $\theta_1 v'(q_1) = c$, where $v'(q_1)$ is the derivative of v with respect to q_1.

Then it chooses (q_2, T_2) to max profit from consumer 2 without violating the incentive compatibility constraint for consumer 2:

$$\max \Pi_2 = T_2 - c * q_2$$
$$s.t. \quad \theta_2 v(q_2) - T_2 \geq \theta_2 v(q_1) - T_1$$

This implies that $T_2 = \theta_2(v(q_2) - v(q_1)) + \theta_1 v(q_1)$ and $\theta_2 v'(q_2) = c$.
The total profit for the monopoly from both consumers are:

$$\Pi = (2\theta_1 - \theta_2)v(q_1) + \theta_2 v(q_2) - c(q_1 + q_2)$$

The derivative of the profit function with respect to q_1 is:

$$\frac{\partial \Pi}{\partial q_1} = (2\theta_1 - \theta_2)v'(q_1) - c < 0$$

which implies that the quantity q_1 is more than optimal (does not maximize profit) and should be reduced.

Problem 6

a) The monopolist would charge each consumer her willingness to pay. Total profits would be:

$$\pi = \$15 + \$10 + \$8 + \$12 = \$45$$

b) There are four possible combinations:

(i) \$15 for sports and \$10 for cooking

(ii) \$8 for sports and \$10 for cooking

(ii) \$15 for sports and \$12 for cooking

(iv) \$8 for sports and \$12 for cooking

Under scheme (i) Type 1 would buy sports and both types subscribe to cooking. Under scheme (ii) Both types would subscribe to both programs.

Under scheme (iii) Type 1 would subscribe to sports and type two to cooking.

Under scheme (iv) both would subscribe to sports and only type 2 would subscribe to cooking. Profits would be

(i) $2 \times 10 + 15 = 35\$$

(ii) $2\,(8 + 10) = 36\$$

(iii) $15 + 12 = 27\$$

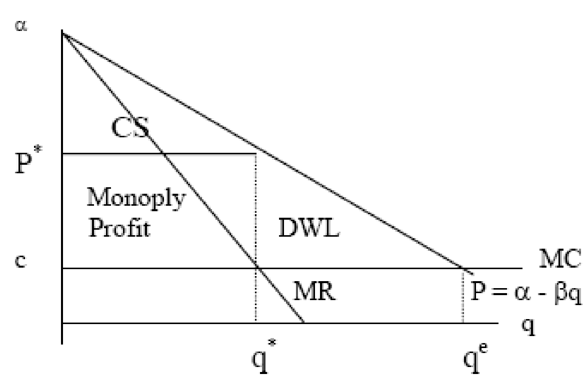
(iv) $2 \times 8 + 16 = 28\$$

Hence, scheme (ii) is the most profitable. The monopolist can obtain a profit of \$40 by only offering both programs

in one package for \$20.

FIGURES

Problem 1(c):



Problem 5(a):