

Solutions to Problem Set 4

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1 High vs. Low Prices

1.1 Finite T

The unique SPE is $\{(L_t)_{t=1}^T, (L_t)_{t=1}^T\}$, or in other words the SPE strategies are "play L in every period" for both players. We need to show that these strategies form a NE in every subgame of the game. Start from the last subgames which correspond to the last period (there are many of these subgames depending on what happened before- e.g. if $T = 2$, there are 4 subgames following the histories (L, L) , (H, L) , (L, H) and (H, H)). In these subgames, the players will now play the stage game one final time. The only NE in these subgames is (L, L) , a NE in dominant strategies. Knowing this, let's move to the second to last subgames and repeat the reasoning. The players are now going to play the stage game once and after that in the following period (the last one) they will pick low prices. Therefore, they are facing the game:

	H	L
L	$11 + 1, 0 + 1$	$1 + 1, 1 + 1$
H	$10 + 1, 10 + 1$	$0 + 1, 15 + 1$

Again the unique NE (in dominant strategies) is (L, L) . Repeating the reasoning we can reach the first period.

1.2 Infinite T , same SPE

Yes. We need to show that the proposed strategies form a NE in every subgame. Suppose the players find themselves at period t . No matter what happened before, if player i 's strategy prescribes to play L from now on, player j 's best response is L from now on as well. Since every subgame looks exactly like the whole game, the strategies "L in every period" are a SPE.

1.3 Infinte T , High Prices

We will show that the following strategies constitute a SPE:

$$s^* = \begin{cases} H, & \text{if } t = 1 \\ H, & \text{in period } t \text{ if } (H, H) \text{ was played in all periods } s < t \\ L, & \text{otherwise} \end{cases}$$

Indeed, we will show that these strategies form a NE in every subgame. There is an infinite number of subgames but we can divide all subgames in two categories: all subgames in which (H, H) has been played so far and all other subgames (i.e. all those that some other action profile appeared before). Let's examine the first category (checking the second category is straightforward and replicates the argument made in part (b)). We will check whether (s^*, s^*) is a NE in these subgames. To do this, we fix player 1's strategy at s^* and check whether s^* is a BR for 2 against s^* (definition of NE). Indeed, if 2 sticks to s^* , he gets:

$$u_2(s^*, s^*) = 10 + \delta 10 + \dots = \frac{10}{1 - \delta}$$

What if he deviates? We might as well look at his best possible deviation. This for sure includes deviating optimally at the first period (of the subgame), since payoffs are discounted (i.e. don't wait some periods and then deviate). The optimal deviation is to pick the low price (here it's the only possible deviation in pure strategies, but in game with say a continuum of actions we must pick the best thing available). After 2 deviates, 1's strategy s^* prescribes low prices forever after. The best response to this (remember, optimal deviation!) is to pick low prices as well. Thus, the deviation gives a payoff of:

$$u_2(s^d, s^*) = 15 + \delta 1 + \dots = 15 + \frac{\delta}{1 - \delta}$$

Therefore, player 2 will stick to s^* if

$$u_2(s^d, s^*) \leq u_2(s^*, s^*)$$

or

$$15 + \frac{\delta}{1 - \delta} \leq \frac{10}{1 - \delta}$$

or

$$\delta \geq \frac{5}{14}$$

Let's do the same thing for player 1 (although, 2's optimal deviation is more profitable so we are really done already):

$$u_1(s^d, s^*) \leq u_1(s^*, s^*)$$

or

$$11 + \frac{\delta}{1 - \delta} \leq \frac{10}{1 - \delta}$$

or

$$\delta \geq \frac{1}{10}$$

Thus we need $5/14$ for each player to obtain a payoff of 10 each period.

Notice that these strategies yield payoff of 10 in every period to both players, if δ satisfies the bound we found. Indeed, we showed that players won't deviate and the threat of low prices forever will never be realized.

1.4 Probability of Ending the Game

Let $\delta = 1 - p$ (See lecture notes), the remaining is the same as in part (c). So $5/14$ is equivalent to $1 - p < 5/14$ or $p < 9/14$.

2 Repeated Choice of Quantity and Collusion

2.1 Cournot Once

Each firm i solves:

$$\max_{q_i} \{(50 - 4(q_i + q_j)) q_i - 2q_i\}$$

The necessary FOC is

$$50 - 8q_i - 4q_j - 2 = 0$$

Impose symmetry: $q_i = q_j$ to get

$$q_i^C = 4, \quad i = 1, 2$$

2.2 One period Collusion

This is the monopolist's problem:

$$\max_{q_i} \{(50 - 4Q) Q - 2Q\}$$

FOC:

$$50 - 4Q - 4Q - 2 = 0$$

or

$$Q = 6$$

Therefore, any pair (q_1, q_2) can maximize joint profit as long as $q_1^m + q_2^m = 6$ (and $q_i \geq 0$).

2.3 Infinitely Repeated Version

Let's call the profits from part (a), π^C and the collusion profits π^m so that

$$\begin{aligned}\pi^C &= 64 \\ \pi^m &= 72\end{aligned}$$

We will show that the strategies:

$$q^* = \begin{cases} q^m, & \text{if } t = 1 \\ q^m, & \text{in period } t \text{ if } (q^m, q^m) \text{ was played in all periods } s < t \\ q^C, & \text{otherwise} \end{cases}$$

are a SPE of the game and sustain collusion. As in problem 1, divide the subgames in two categories and let's examine the subgames where monopoly quantities have been observed so far. Fix i 's strategy to q^* . What is j 's optimal deviation? In the first period of the subgame, player i chooses q_i^m . Let's suppose that $q_i^m = q_j^m = 3$ (but the argument works for any other). What is j 's optimal one-shot deviation?

$$\max_{q_j} \{(50 - 4(3 + q_j))q_j - 2q_j\}$$

so that

$$q_j^d = 4.5$$

and the profit j gets is

$$\pi^d = 4.5(50 - 4(4.5) - 4(3) - 2) = 81$$

After this deviation, i 's strategy prescribes q^C and the BR to that is q^C as well. Therefore, it must be that:

$$81 + \frac{64\delta}{1 - \delta} \leq \frac{72}{1 - \delta}$$

so that

$$\delta \geq \frac{9}{17}$$

Since

$$\delta = \frac{1}{1 + 0.05}$$

we get that

$$\frac{1}{1 + 0.05} \geq \frac{9}{17}$$

which holds.

3 Duopoly Collusion

3.1 Discount every period

Same reasoning as above: First solve the monopoly problem to get the collusive profits:

$$\max_{q_i} \{(50 - Q/2) Q\}$$

FOC:

$$50 - Q/2 - Q/2 = 0$$

or

$$Q = 50$$

Therefore, any pair (q_1, q_2) can maximize joint profit as long as $q_1^m + q_2^m = 50$ (and $q_i \geq 0$) and $\pi^m = 1250$ (which they split in half).

Now solve the Cournot problem to find the threat payoffs. Each firm i solves:

$$\max_{q_i} \{(50 - (q_i + q_j)/2) q_i\}$$

The necessary FOC is

$$50 - q_i - q_j/2 = 0$$

Impose symmetry: $q_i = q_j$ to get

$$q_i^C = 50/3, \quad i = 1, 2$$

and

$$\pi^C = 5000/9 = 555.55$$

To find the optimal deviation, we need to find the BR to q_i^m . Assume that $q_i^m = 25$ so that the optimal deviation solves

$$\max_{q_i} (50 - q_i/2 - 25/2)$$

So that

$$q_i^d = 75/2$$

and profit is

$$\pi^d = 703.1$$

Therefore, to sustain the SPE with the same strategies as in problem 2, it must be:

$$703.1 + \frac{555.55\delta}{1 - \delta} \leq \frac{1250/2}{1 - \delta}$$

or

$$\delta \geq 0.53$$

3.2 Produce in Odd Periods

Consider the same strategies. Cooperation provides payoff:

$$\pi^m + \delta^2 \pi^m + \delta^4 \pi^m + \dots = \frac{\pi^m}{(1 - \delta^2)}$$

While the optimal deviation gives payoff:

$$\pi^d + \delta^2 \pi^C + \delta^4 \pi^C + \dots = \pi^d + \frac{\delta^2 \pi^C}{(1 - \delta^2)}$$

So that the strategies form a SPE if:

$$\frac{\pi^m}{(1 - \delta^2)} \geq \pi^d + \frac{\delta^2 \pi^C}{(1 - \delta^2)}$$

or

$$\delta \geq 0.7276$$

(Remember we also need to check that the strategies are a NE in subgames where profiles other than (q^*, q^*) have been observed, but this is just Nash behavior in every period).

Intuition: Now it is harder to sustain collusion because payoff from deviation lasts longer and punishment is delayed.