

Partial Pooling at the Reserve Price in Auctions with Resale Opportunities

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Two features common to many auctions are the use of reserve prices and the existence of secondary markets for the goods being sold. Even in simple symmetric settings, the combination of these features can preclude existence of an equilibrium in symmetric separating bidding strategies. With a reserve price sufficiently far below the maximum (endogenous) valuation, a symmetric equilibrium still exists, but with some types pooling at the reserve. The optimal reserve price depends not only on the joint distribution of bidders' information before and after the auction, but also on how surplus is divided in the secondary market. *Journal of Economic Literature* Classification Numbers: D44, C7, L1, D82 © 2000 Academic Press

1. INTRODUCTION

Two features common to many auctions are the use of reserve prices and the existence of resale markets. Existing theory offers compelling support for the use of reserve prices by profit maximizing sellers. With symmetric independent private values, a standard auction with an optimal reserve price is optimal among all possible selling mechanisms (Myerson, 1981). Even outside the independent private values framework, a (potentially) binding reserve price can often be used by a seller to increase her expected revenues. However, the literature on optimal auctions in general and that on the use of reserve prices in particular has focused on models that ignore resale opportunities. While the auction literature has long acknowledged the prevalence of resale markets in practice, the effects of resale opportunities on auctions have received little careful attention. Milgrom (1987)

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was the first to model auctions with resale. Subsequent papers have shown that a resale opportunity can have fundamental effects on auctions, with significant implications for bidding strategies, seller behavior, and interpretation of bidding data (Bikhchandani and Huang, 1989; Gupta and Lebrun, 1997; Haile, 1996, 1999, 2000). This paper builds on that work by considering the effects of binding reserve prices when each bidder places positive probability on buying in a secondary market if he loses the auction.

A central paper in the auction literature is Milgrom and Weber's (1982) analysis of symmetric "affiliated values" models. In that framework, a reserve price has only minor impacts on the form of equilibrium bidding strategies. Here, despite some strong similarities to the affiliated values framework, a binding reserve price precludes existence of an equilibrium in separating bidding strategies. The intuition is straightforward. Consider a second-price sealed bid auction.¹ In a separating equilibrium, a bidder must participate if winning would give him an expected payoff (gross of the price paid) of at least the reserve price—otherwise he could deviate by bidding the reserve and winning when the object would otherwise have gone unsold. Once the participation decision is made, however, the effect of a marginal change in a player's bid on his payoff is zero except when an opponent is making the same bid he is. Therefore, every bidder offering more than the reserve price must choose his bid based on an assumption that the object will still be sold if he loses the auction. Bidders are then willing to pay a price equal to the difference between the expected gross payoff conditional on winning the auction and the (strictly positive, due to the resale market) expected payoff conditional on losing. For types just above the participation margin, this difference must be less than the reserve price, contradicting the requirement that these types bid more than the reserve.

This negative result is itself important given the almost exclusive focus on separating equilibria in the theoretical and empirical auction literature. While there are other examples in which an equilibrium in strictly increasing bidding strategies fails to exist, here this is the case in extremely simple symmetric settings, with the result driven by features present in many applications. This result, of course, begs the question of what sort of bidding we should expect to observe in such situations. Answering this question is important for understanding bidding strategies and interpreting bidding data in many applications where binding reserve prices are used. A natural intuition might suggest that for bidders just above the participation margin, the constraint that bids be at least as high as the reserve price will bind, forcing them all to bid the reserve. While this intuition is incomplete,

¹The argument is essentially the same for other standard auctions.

the qualitative description of the equilibrium (when it exists) is correct: an interval of types pooling at the reserve price with bids strictly increasing in types otherwise.² However, if the reserve price is too close to the maximum possible valuation, no symmetric equilibrium will exist.

The following section presents the model. For simplicity, I study a second-price sealed bid auction and model outcomes in the resale market using the Nash bargaining solution. Neither of these simplifications is essential to the results. However, a key assumption is that bidders have noisy signals of their private values at the time of the auction. This sort of uncertainty has no effect on standard (one-shot) auction models with risk neutral bidders. Here it causes bidders to place positive probability on there being gains to trade in the resale market once the uncertainty is resolved, even after losing an auction in which all bidders use the same monotonic bidding strategy. This is only one possible structure motivating an active resale market, but one that describes many applications: bidders for an antique may be uncertain how it will look in their homes but will find out once they purchase; bidders for a transferable operating license may face idiosyncratic technological uncertainty that will be resolved over time or may be unsure which complementary licenses they will obtain; bidders for timber harvesting contracts may face idiosyncratic demand uncertainty that will be resolved before the harvest deadline (Haile, 2000). Indeed, some uncertainty seems likely in almost any application, and an arbitrarily small amount is enough to give the results.

After setting up the model, Section 2 presents the nonexistence result and the derivation of the partially pooling equilibrium. Section 3 then discusses implications for the seller's optimal reserve price. Section 4 concludes. Proofs of existence and uniqueness of the partially pooling equilibrium are given in an appendix.

2. THE MODEL

Consider a two-stage game between two risk neutral buyers in which an auction is followed by bargaining in a secondary market. In the first

²Other models leading to such partial pooling equilibria (although for quite different reasons) include Matthews (1984) and Hendricks and Porter (1988). Independently, Jehiel and Moldovanu (2000) obtain such an equilibrium in a related model in which awarding the object to one bidder imposes an externality on the others. A (positive) externality of this sort arises in the model of auctions with resale here, giving losing bidders positive expected payoffs. Unlike their model, a positive externality is also imposed on the auction winner here, since the option value of selling in the resale market is positive. Elsewhere (Haile, 1996, 1999, 2000) I have referred to the effect of these externalities on bidders' (endogenous) valuations as the *resale buyer effect* and *resale seller effect*, respectively.

stage, a second-price sealed bid auction is held with a reserve price r . If no bids above r are received, the seller keeps the object and the game ends. Ties in the bidding are resolved by the toss of a fair coin. At the time of the auction, each bidder $i \in \{1, 2\}$ has only a noisy signal X_i of his “use value” U_i , which is the value he places on consuming the object himself. In the second stage, information is complete (both players’ use values are publicly observed) and any gains to resale trade are divided according to the symmetric Nash bargaining solution.

Each X_i has unconditional distribution $F(\cdot)$ with associated density $f(\cdot)$ and support $[0, 1]$. The conditional distribution of U_i given X_i is $G(\cdot|X_i)$. While signals are independent of each other, each X_i and U_i are affiliated, implying that $G(u|x)$ is nonincreasing in x . To ensure that signals are informative, I further assume that whenever $\hat{x} > x$, the distribution $G(\cdot|\hat{x})$ strictly dominates $G(\cdot|x)$. I assume $\int_{-\infty}^{\infty} G(u|y) dG(u|x) < 1$ for all x and y , implying a positive probability of gains to resale trade between the bidders regardless of their first-stage signals or the allocation of the object. Finally, for simplicity I assume that given any differentiable function $\varphi(\cdot) : \mathfrak{R} \rightarrow \mathfrak{R}$, $E[\varphi(U_i)|X_i = x]$ is differentiable with respect to x .

A bidder i who wins the first-stage auction may keep the object himself, but places positive probability on selling in the resale market. Given each of the signals, his expected payoff (gross of the price paid to the initial seller) when he wins the auction is $w(X_i, X_{-i})$, where

$$w(x, y) = \int_{-\infty}^{\infty} \left[u + \int_u^{\infty} \left(\frac{t-u}{2} \right) dG(t|y) \right] dG(u|x).$$

The first term inside the brackets is the bidder’s use value and the second term gives his share of the expected gains to resale trade conditional on this use value. Similarly, the expected payoff to bidder i when he loses is $\ell(X_i, X_{-i})$, with

$$\ell(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^u \left(\frac{u-t}{2} \right) dG(t|y) dG(u|x).$$

Define

$$v(X_i, X_{-i}) = w(X_i, X_{-i}) - \ell(X_i, X_{-i}).$$

This function gives the expected value to bidder i of winning the auction rather than losing, conditional on the first-stage signals. This is i ’s willingness to pay at the auction given the realizations of X_i and X_{-i} , i.e., his “valuation.” It is straightforward to confirm that both $w(x, y)$ and $v(x, y)$ are differentiable and strictly increasing in x and y .

With this notation, it is possible to formalize the assumption that the reserve price binds with positive probability. I assume the reserve price is higher than the lowest possible valuation:

$$r > v(0, 0).$$

We will see below that if the reserve price is above the highest possible valuation, the only possible symmetric equilibrium is one which no player ever bids. I therefore focus on the case $r \in (v(0, 0), v(1, 1))$.

I focus initially on symmetric Bayesian Nash equilibria in strictly increasing bidding strategies. Suppose there is such an equilibrium, with $b(\cdot)$ the equilibrium bid function. A bidder with signal x who submits a bid of at least the reserve price solves³

$$\begin{aligned} \max_{\tilde{x} \in [0, 1]: b(\tilde{x}) \geq r} \pi(x, \tilde{x}) &= \int_0^{\tilde{x}} [w(x, y) - \max\{b(y), r\}] dF(y) \\ &+ \int_{\tilde{x}}^1 \ell(x, y) dF(y). \end{aligned} \quad (1)$$

The first-order condition implies that in any equilibrium, each player who bids at least r follows the bid function

$$b(x) = v(x, x). \quad (2)$$

It is straightforward to confirm that $\partial \pi(x, \tilde{x}) / \partial \tilde{x}$ takes the same sign as $(x - \tilde{x})$, ensuring that (2) characterizes an optimum.

Although the valuation $v(x, x)$ is endogenous here, (2) suggests the familiar result that in the symmetric equilibrium of a second-price auction, each player bids his valuation, conditional on an assumption that his most competitive opponent has the same type he does. However, this strategy does not constitute an equilibrium. The failure occurs at the participation margin. To see this, suppose all bidders bidding at least r followed (2) in equilibrium and let $x_0(r)$ be the lowest type willing to bid at least r . A bid of r wins only when no other bid is submitted, so in equilibrium the gain to a type x bidder from bidding r rather than submitting no bid is

$$F(x_0(r)) \left[E[w(x, y) | y \leq x_0(r)] - r \right].$$

This implies

$$x_0(r) = \inf \left\{ x \in [0, 1] : E[w(x, y) | y \leq x] \geq r \right\}.$$

³Since ties occur with probability zero, we can restrict attention to bids in the range of $b(\cdot)$ without loss of generality.

Since $w(x, y)$ is continuous and strictly increasing in x and y , we must have either $x_0(r) = 0$ or

$$E[w(x_0(r), y) | y \leq x_0(r)] = r. \quad (3)$$

Now define $\hat{x}(r)$ by the equation

$$v(\hat{x}(r), \hat{x}(r)) = r.$$

Since all types $x > x_0(r)$ must submit bids above r in equilibrium, the first-order condition (2) implies that we must have $x_0(r) \geq \hat{x}(r)$. The following lemma shows that $x_0(r) < \hat{x}(r)$, giving a contradiction.

LEMMA 1. For all $r \in (v(0, 0), v(1, 1)]$, $\hat{x}(r) > x_0(r)$.

Proof. If $x_0(r) = 0$, $\hat{x}(r) > x_0(r)$ follows immediately from the definition of \hat{x} and the assumption $r > v(0, 0)$. So assume $x_0 > 0$. From (3) and the definition of $\hat{x}(r)$ we know

$$\int_0^{x_0(r)} w(x_0(r), y) \frac{f(y)}{F(x_0(r))} dy = r = v(\hat{x}(r), \hat{x}(r)).$$

Suppose $x_0(r) \geq \hat{x}(r)$. Then it must be the case that

$$\int_0^{x_0(r)} w(x_0(r), y) \frac{f(y)}{F(x_0(r))} dy \leq v(x_0(r), x_0(r)). \quad (4)$$

Since $w(x, y)$ is equal to

$$E[U_i | X_i = x] + E\left[\frac{U_{-i} - U_i}{2} \mid X_i = x, X_{-i} = y, U_{-i} > U_i\right] \\ \times \Pr(U_{-i} > U_i | X_i = x, X_{-i} = y) \quad (5)$$

and $\ell(x, x)$ is equal to

$$E\left[\frac{U_i - U_{-i}}{2} \mid X_i = X_{-i} = x, U_i > U_{-i}\right] \Pr(U_i > U_{-i} | X_i = X_{-i} = x),$$

symmetry implies

$$v(x, x) = E[U_i | X_i = x].$$

The assumption that gains to trade exist with positive probability given any pair of first-stage signals ensures that the second term in (5) is strictly positive, implying $w(x, y) > v(x, x)$ for all x and y , contradicting (4). ■

This proves the following theorem:⁴

THEOREM 1. There exists no equilibrium in symmetric strictly increasing bidding strategies.

⁴With minor modifications, the same argument shows that there is also no asymmetric equilibrium in strictly increasing bidding strategies.

Some intuition can be obtained by comparing the auction with resale to the seemingly similar second-price sealed bid auction with affiliated values and no resale market (Milgrom and Weber, 1982). The analysis above easily incorporates the no-resale case: the objective function (3), first-order condition (4), and participation condition (5) are all still valid in this case, with two differences. First, without resale $\ell(x, y) \equiv 0$ for all x, y . Second, $v(x, y) = w(x, y) = E[V_i | X_i = x, X_{-i} = y]$, where V_i is the value of the object to bidder i . The first difference is the important one, since it implies that without resale the value a bidder places on winning the auction does not depend on whether the object is sold when he loses. Hence, the marginal type's bid in the Milgrom–Weber model is

$$b(x_0(r)) = w(x_0(r), x_0(r)) \geq E[w(x_0(r), y) | y \leq x_0(r)] = r.$$

This gap between the reserve price and the lowest equilibrium bid results from the difference between conditioning on $y = x_0(r)$ and conditioning on $y \leq x_0(r)$ when calculating the expected value of winning the auction. Now consider the same difference in the model with resale. When a bidder with type $x_0(r)$ conditions on $y \leq x_0(r)$, he believes with probability one that the object will go unsold if he does not bid. Bidders receive payoffs of zero when the object goes unsold, so the marginal type is willing to pay $E[w(x_0(r), y) | y \leq x_0(r)]$ to win. When the marginal bidder conditions on $y = x_0(r)$ this implies not only a higher realization of y , but also that the object would still be sold if he deviated by making no bid. This is important because it means he could then buy in the resale market, implying that his maximum willingness to pay at the auction is $w(x_0(r), x_0(r)) - \ell(x_0(r), x_0(r))$. Lemma 1 shows that this is less than the reserve price; i.e., the inequality creating the “gap” in the Milgrom–Weber model is reversed:

$$b(x_0(r)) = v(x_0(r), x_0(r)) < r.$$

Hence, types just above $x_0(r)$ are unwilling to bid more than the reserve price as part of a symmetric separating equilibrium—they could profit by deviating to a bid closer to r and winning almost only when the object would otherwise have gone unsold. However, these same types prefer paying r for the object to letting it go unsold. This conflict precludes existence of a symmetric separating equilibrium.

This nonexistence result is important given the focus on symmetric separating equilibria in the theoretical and empirical auction literature. However, the obvious question this raises is what sort of equilibria, if any, do exist. Using standard arguments, it is straightforward to show that in any symmetric equilibrium, bids must be strictly increasing in types for all types bidding more than the reserve price. This leaves only one possible sort of symmetric equilibrium: one in which a positive mass of types bid the reserve

price, with bidding strictly increasing above the reserve. The following result shows that it is possible to derive exactly one such equilibrium as long as the reserve price is not too high.

THEOREM 2. *If $r \leq \int_{x_0(r)}^1 v(1, y)f(y)/(1 - F(x_0(r)))dy$, there exists a unique symmetric equilibrium: all types in an interval $[\underline{x}, \bar{x}] \subset [0, 1]$ bid the reserve price, with $x_0(r) \leq \underline{x} < \hat{x}(r) < \bar{x}$. Above \bar{x} , bids strictly increase in types, following the bid function $b(x) = v(x, x)$.*

Proof. Suppose there is a partially pooling equilibrium in which bids are strictly increasing above the reserve price but all types in $[\underline{x}, \bar{x}] \subset [0, 1]$ bid r . The optimization problem (1) for any bidder offering more than r is unaffected by the pooling at the reserve price. Hence, the only question is whether there is an interval $[\underline{x}, \bar{x}]$ over which it is optimal for all types to bid r while all types above \bar{x} bid more than r . Incentive compatibility requires that bidders with signals $x \in [\underline{x}, \bar{x}]$ prefer bidding r to submitting no bid; i.e.,

$$\int_0^{\underline{x}} [w(x, y) - r] dF(y) + \int_{\bar{x}}^1 \ell(x, y) dF(y) + \int_{\underline{x}}^{\bar{x}} \frac{1}{2} [w(x, y) - r + \ell(x, y)] dF(y) \geq \int_{\underline{x}}^1 \ell(x, y) dF(y).$$

This inequality must be reversed for all $x < \underline{x}$. Since both $w(x, y)$ and $v(x, y)$ are continuous and strictly increasing in x , this implies

$$\int_0^{\underline{x}} [w(\underline{x}, y) - r] dF(y) + \frac{1}{2} \int_{\underline{x}}^{\bar{x}} [v(\underline{x}, y) - r] dF(y) = 0. \quad (6)$$

This gives the first of two incentive compatibility constraints that a solution (\underline{x}, \bar{x}) must satisfy. To obtain the second, note that all bidders with signals $x \geq \bar{x}$ must prefer bidding $b(x) > r$ to bidding r :

$$\int_0^{\bar{x}} [w(x, y) - r] dF(y) + \int_{\bar{x}}^x [w(x, y) - b(y)] dF(y) + \int_x^1 \ell(x, y) dF(y) \geq \int_0^{\underline{x}} [w(x, y) - r] dF(y) + \int_{\bar{x}}^1 \ell(x, y) dF(y) + \frac{1}{2} \int_{\underline{x}}^{\bar{x}} [w(x, y) - r + \ell(x, y)] dF(y).$$

This reduces to

$$\frac{1}{2} \int_{\underline{x}}^{\bar{x}} [v(x, y) - r] dF(y) + \int_{\bar{x}}^x [v(x, y) - v(y, y)] dF(y) \geq 0 \quad \forall x \geq \bar{x}. \quad (7)$$

Meanwhile, types in $[\underline{x}, \bar{x}]$ must prefer a bid of r to any higher bid; i.e, for any $\tilde{x} \in [\underline{x}, \bar{x}]$

$$\frac{1}{2} \int_{\underline{x}}^{\tilde{x}} [v(\tilde{x}, y) - r] dF(y) + \int_{\bar{x}}^x [v(\tilde{x}, y) - v(y, y)] dF(y) \leq 0 \quad \forall x \geq \bar{x}. \quad (8)$$

Since the left side of (7) increases in x and the left side of (8) decreases in x , it is necessary and sufficient that both inequalities hold for $x = \bar{x}$. Then, since the left side of (8) increases in \bar{x} , it is necessary and sufficient that this inequality hold for $\bar{x} = \bar{x}$. This gives

$$\int_{\underline{x}}^{\bar{x}} [v(\bar{x}, y) - r] dF(y) = 0. \quad (9)$$

The hypothesis of the theorem requires that this equation have a solution for $\bar{x} \leq 1$ when $\underline{x} = x_0(r)$.

One can confirm that when Eqs. (6) and (9) hold, types above \bar{x} prefer bidding $b(x)$ to submitting no bid, while types below \underline{x} prefer not bidding to any bid above r . Hence these equations simultaneously determine the combination(s) of \underline{x} and \bar{x} that are part of a partially pooling equilibrium. Equation (6) requires that all types above \underline{x} (and only these) prefer bidding r to submitting no bid. Equation (9) requires that all types above \bar{x} (and only these) are willing to “capture the atom” by bidding strictly above r . Rewrite Eqs. (6) and (9) as

$$Q(\underline{x}, \bar{x}) = 0 \quad (6')$$

$$R(\underline{x}, \bar{x}) = 0. \quad (9')$$

In the appendix it is shown that these two equations have a unique solution under the hypothesis of the theorem, and that the interval $[\underline{x}, \bar{x}]$ lies to the right of $[x_0(r), \hat{x}(r)]$ as claimed. ■

We can think of the failed existence of a fully separating equilibrium as resulting from a gap between type $\hat{x}(r)$, for whom the constraint not to bid more than r in a separating equilibrium binds, and the type $x_0(r)$, for whom the constraint not to bid less than r binds. Having types in $(x_0(r), \hat{x}(r))$ bid the reserve price doesn't yield an equilibrium since the pooling changes the incentive compatibility constraints. However, by appropriately shifting the interval of types who pool at r to account for this, an equilibrium can be obtained. Consider the lower participation margin. With partial pooling at r , winning with a bid of r no longer implies that one has won when the object would otherwise have gone unsold. Therefore, even the lowest type to bid must account for the opportunity to buy in the secondary market when determining his willingness to pay. Consequently, this marginal type \underline{x} must be above $x_0(r)$. At the upper margin of the pooling interval, consider a type just above $\hat{x}(r)$. A deviation to a bid of r is more attractive when types below $\hat{x}(r)$ also bid r , since a bid of r still wins when the object would otherwise have gone unsold but doesn't always win when one's opponent's type is in $(\underline{x}, \hat{x}(r))$. When $y \in (\underline{x}, \hat{x}(r))$, $v(\hat{x}(r), y) < r$, so types just above $\hat{x}(r)$ prefer to lose to another bidder rather than win at a price of r . By joining the pooling at r , these types accomplish this with positive probability, giving $\bar{x} > \hat{x}(r)$.

EXAMPLE. Suppose each X_i is distributed uniformly on $[0, 1]$ and that U_i equals X_i with probability $1/2$ and 0 with probability $1/2$. In this case $w(x, y) = x/2 + y/8$ and $v(x, y) = (x + y)/4$ for all x, y . Suppose the seller sets a reserve price of $1/4$ (the optimal reserve price without resale). Figure 1 illustrates the two incentive compatibility constraints and the unique solution $(\underline{x}^*, \bar{x}^*) \approx (0.447, 0.518)$. Higher reserve prices would shift the curves defined by $Q(\underline{x}, \bar{x}) = 0$ and $R(\underline{x}, \bar{x}) = 0$ to the right. For reserve prices sufficiently high (above about 0.483) there will be no intersection of the two curves inside $[0, 1] \times [0, 1]$, and even the partial pooling equilibrium will fail to exist. Thus the reserve price must be sufficiently far below $v(1, 1)$ (equal to $1/2$ here) for any symmetric equilibrium to exist. \diamond

Since $\bar{x} > \hat{x}(r)$, we must have $\hat{x}(r) < 1$ (i.e., $r < v(1, 1)$) for a partial pooling equilibrium to exist. However, the statement of Theorem 2 relies on a stronger sufficient condition. In the example, r must lie *sufficiently* far below $v(1, 1)$ for the equilibrium to exist. Theorem 3 shows that this is true in general.

THEOREM 3. *There exists $\epsilon > 0$ such that for any $r \in (v(1, 1) - \epsilon, E[w(1, y)])$ there exists no symmetric equilibrium.*

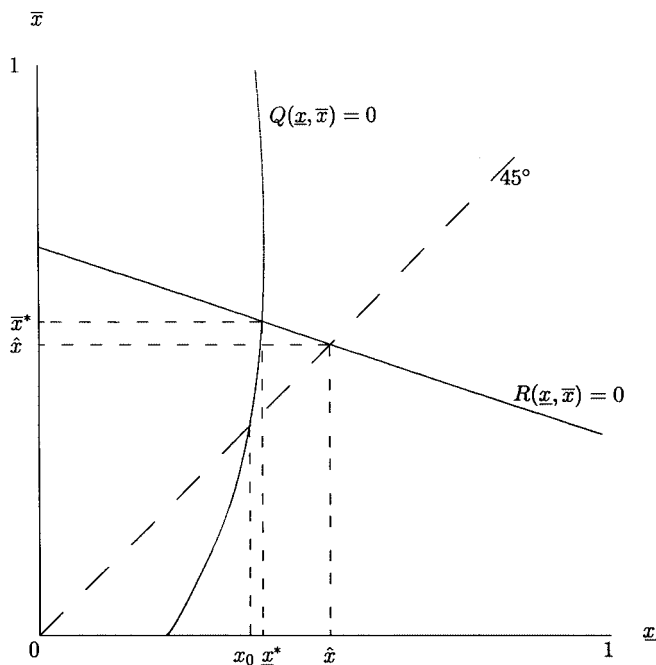


FIGURE 1

Proof. The proof of Lemma 1 showed that $E[w(x, y)|y \leq x] > v(x, x) \forall x, y$, ensuring that $E[w(1, y)] > v(1, 1)$. For $r \in (v(1, 1), E[w(1, y)])$ it is clear from (2) and the fact that $w(x, x) > v(x, x)$ that no symmetric equilibrium can exist. (For $r \geq E[w(1, y)]$ there is a trivial symmetric equilibrium in which no player ever bids.) So suppose that for all $r \leq v(1, 1)$ a symmetric equilibrium exists. The analysis above implies that this must be a partial pooling equilibrium, with the interval of pooling types $[\underline{x}, \bar{x}]$ characterized by Eqs. (6) and (9). But consider what these equations imply as $r \rightarrow v(1, 1)$. First, since \bar{x} cannot be greater than one, (9) and the fact that $v(x, y)$ strictly increases in y imply that $\underline{x} \rightarrow 1$. This implies that $\bar{x} \rightarrow 1$. Thus, the second term in (6) converges to zero, so the first term must converge to zero as well. This requires that $\underline{x} \rightarrow x_0(v(1, 1))$. Thus, since \underline{x} converges to both 1 and $x_0(v(1, 1))$, these must be equal. Since $\hat{x}(v(1, 1)) = 1$ by definition, this contradicts Lemma 1. ■

3. OPTIMAL RESERVE PRICES

If there were no binding reserve price, the first-order condition (2) would characterize equilibrium bidding for all bidders (Haile, 1999): all would bid exactly as if they had independent private values

$$V_i = v(X_i, X_i)$$

with distribution $H(\cdot)$ defined by

$$H(v(X_i, X_i)) = F(X_i)$$

in an auction without resale. This might suggest that an optimal reserve price could be determined by the equation

$$r^* = \frac{1 - H(r^*)}{h(r^*)} \quad (10)$$

following standard results for independent private value auctions without resale (Myerson, 1981; Riley and Samuelson, 1981). Partial pooling at the reserve price implies this is not the case.

Assuming it is not optimal to set a reserve price so high that no symmetric equilibrium exists,⁵ a seller chooses r to maximize

$$r \left[2(1 - F(\underline{x}))F(\bar{x}) - (F(\bar{x}) - F(\underline{x}))^2 \right] + \int_{\underline{x}}^1 \int_{\bar{x}}^t v(s, s) 2f(s)f(t) ds dt$$

⁵The preceding analysis provides no prediction regarding bidder behavior when a higher reserve price is set. Empirical evidence suggesting that actual reserve prices are far below levels which would be optimal for auctions without resale (e.g., Li et al., 1997; Ma, 1998; McAfee et al., 1995; McAfee and Vincent, 1992; Paarsch, 1997) provides some confidence regarding the descriptive value of the partial pooling equilibrium. Nonetheless, the optimality of a reserve price that precludes existence of a symmetric equilibrium cannot be ruled out.

where \underline{x} and \bar{x} are determined by the choice of r , and can therefore be written $\underline{x}(r)$ and $\bar{x}(r)$. Since the Jacobian matrix

$$\begin{bmatrix} Q_1(\underline{x}, \bar{x}) & Q_2(\underline{x}, \bar{x}) \\ R_1(\underline{x}, \bar{x}) & R_2(\underline{x}, \bar{x}) \end{bmatrix}$$

is nonsingular at any solution to (6) and (9) (this is shown in the appendix), the implicit function theorem ensures that the functions $\underline{x}(\cdot)$ and $\bar{x}(\cdot)$ are well defined and differentiable in a neighborhood of any solution. The first-order condition for the seller's problem is

$$2F(\bar{x}) - F(\bar{x})^2 - F(\underline{x})^2 + 2\bar{x}'(r)f(\bar{x})\left[r - v(\bar{x}, \bar{x})\right]\left[1 - F(\bar{x})\right] - 2\underline{x}'(r)rf(\underline{x})F(\underline{x}) = 0. \quad (11)$$

Identifying an optimal reserve price is then achieved by simultaneously solving Eqs. (6), (9), and (11).⁶

EXAMPLE (Continued). Solving Eqs. (6), (9), and (11) for the example above yields

$$\begin{pmatrix} r^* \\ \underline{x}(r^*) \\ \bar{x}(r^*) \end{pmatrix} = \begin{pmatrix} 0.2943 \\ 0.5262 \\ 0.6095 \end{pmatrix}.$$

The seller's expected revenue with the optimal reserve is approximately 0.2244, which dominates the expected revenues of 0.1667 that would be obtained without a binding reserve. Note that the optimal reserve price is higher than that which would be optimal without resale (0.25). This is not the result of a net increase in the value of winning the auction due to the resale opportunity. In fact, Eq. (10) would indicate that a reserve price of 0.25 is still optimal.⁷ The derivation of (10), however, depends on an assumption that only one type, $\hat{x} = 0.5$, will bid the reserve price. With a reserve price of 0.25, all bidders in an interval of approximately (0.447, 0.518) bid the reserve price. Optimality is determined by the trade-off between changes in the revenue obtained when the reserve price binds and the probability that at least one bidder is willing to pay the reserve. If all pooling types were below 0.5, the pooling would raise the probability that at least one player bids, relative to that when only types above \hat{x} bid. This reduction in the marginal cost of an increase in r would clearly lead to a higher optimal reserve. While types above 0.5 do pool, the mass of these

⁶The second-order condition is quite complicated and is not verified here, although optimality of the solution defined by (11) has been verified numerically in examples. Note that if (11) has multiple roots, each must be checked for optimality.

⁷This is due to the fact that $v(x, x) = E[U_i|X_i = x]$ given this particular specification of the resale market. In general bidder valuations with a resale opportunity can be higher or lower than those in the same auction without resale. See Haile (1999) for examples.

types is much smaller than that of the pooling types below 0.5, so it is still optimal to raise the reserve price above 0.25. \diamond

Unlike (10), (11) obviously is not equivalent to the condition defining an optimal ultimatum offer to a single buyer. The same is, of course, true in standard common value auctions, due to the correlation of bidders' types. Here types are independent, but the more complex equilibrium structure makes the seller's problem considerably more demanding. Setting a reserve price that is optimal conditional on the complete ex ante specification of the game (the usual notion of optimality) requires a great deal of information—not only the distribution $F(\cdot)$, but also the function $v(\cdot, \cdot)$, which will vary with the structure of the resale market, the number of bidders, and the stochastic relationship between bidders' signals and use values.

4. CONCLUSION

This paper has presented a simple model of auctions with resale opportunities in order to illustrate the effects of binding reserve prices on equilibrium bidding. The focus has been on a model in which each bidder places positive probability on there being gains to trade in the resale market, even after losing the auction to an opponent whose expected value for the object was greater than his own. While no fully separating equilibrium exists, bidding may follow strategies that are similar to those in the fully separating symmetric equilibrium of the same auction without a reserve price, but with an interval of bidder types pooling at the reserve.

These results are important for understanding bidding at a large share of real-world auctions, where uncertainty over values, reserve prices, and resale opportunities are common. Furthermore, there are many cases in which legal restrictions or practical considerations may limit sellers to standard auction mechanisms, leaving the reserve price as the primary policy tool. Hence identifying the information needed to set optimal reserve prices in such settings is important.

The option values of buying and selling in the resale market are the key sources of the more complex valuation and bidding structure here and clearly will affect buyer strategies in any selling mechanism. Hence, while many important questions remain open, this analysis sheds light on the issues that must be dealt with in the broader problem of mechanism design in markets with resale opportunities.

APPENDIX

In this appendix the proof of Theorem 2 is completed by showing that there exists a unique solution to Eqs. (6) and (9) under the hypothesis of

the theorem and that this solution satisfies

$$x_0(r) \leq \underline{x} < \hat{x}(r) < \bar{x}. \quad (12)$$

A few lemmas are needed to show that Brouwer's fixed point theorem can be applied to prove existence. A simple argument using the implicit function theorem then ensures uniqueness.

To verify (12), first note that if $\underline{x} < x_0(r)$, the first term in (6) is negative. Thus, the second term in (6) must be strictly positive, implying that (9) cannot hold. If $\underline{x} \geq \hat{x}(r)$, then since $\hat{x}(r) > x_0(r)$ (Lemma 1) the left side of (6) would be strictly positive. Finally, if $\underline{x} < \bar{x} \leq \hat{x}(r)$

$$\begin{aligned} R(\underline{x}, \bar{x}) &= \int_{\underline{x}}^{\bar{x}} [v(\bar{x}, y) - r] dF(y) \\ &< \int_{\underline{x}}^{\bar{x}} [v(\hat{x}(r), \hat{x}(r)) - r] dF(y) = 0 \end{aligned}$$

contradicting (9). Hence, we must have $\bar{x} > \hat{x}(r)$.

The definitions of $x_0(r)$ and $\hat{x}(r)$ and the fact that both $w(x, y)$ and $v(x, y)$ strictly increase in x imply that the partial derivative

$$\begin{aligned} Q_1(\underline{x}, \bar{x}) &= [w(\underline{x}, \underline{x}) - r]f(\underline{x}) + \int_0^{\underline{x}} w_1(\underline{x}, y) dF(y) \\ &\quad - \frac{1}{2}[v(\underline{x}, \underline{x}) - r]f(\underline{x}) + \frac{1}{2} \int_{\underline{x}}^{\bar{x}} v_1(\underline{x}, y) dF(y) \end{aligned}$$

is positive for $\underline{x} \in [x_0(r), \hat{x}(r)]$. Although

$$Q_2(\underline{x}, \bar{x}) = \frac{1}{2}[v(\underline{x}, \bar{x}) - r]f(\bar{x})$$

has an indeterminate sign in general, at any solution to (6) and (9)

$$\begin{aligned} Q_2(\underline{x}, \bar{x}) &= \frac{1}{2}[v(\underline{x}, \bar{x}) - r]f(\bar{x}) \\ &= \frac{1}{2}[v(\bar{x}, \underline{x}) - r]f(\bar{x}) \\ &< \frac{1}{2} \frac{f(\bar{x})}{F(\bar{x}) - F(\underline{x})} \int_{\underline{x}}^{\bar{x}} [v(\bar{x}, y) - r] dF(y) \\ &= 0. \end{aligned}$$

Here, the second equality follows from the fact (easily confirmed) that $v(x, y) = v(y, x) \forall x, y$; the inequality follows from the fact that $v_2(x, y) > 0 \forall x, y$; the last equality follows from (9).

The partial derivative

$$R_2(\underline{x}, \bar{x}) = [v(\bar{x}, \bar{x}) - r]f(\bar{x}) + \int_{\underline{x}}^{\bar{x}} v_1(\bar{x}, y) dF(y)$$

is positive for all $\bar{x} \geq \hat{x}(r)$, while the sign of

$$R_1(\underline{x}, \bar{x}) = -[v(\bar{x}, \underline{x}) - r]f(\underline{x})$$

is positive for values of \underline{x} and \bar{x} satisfying (9). These results are used in the lemmas below and also imply the nonsingularity of the Jacobian matrix referred to in the discussion of the optimal reserve price.

LEMMA 2. $\bar{x}_R(x) \equiv \{\bar{x} > x : R(x, \bar{x}) = 0\}$ defines a continuous decreasing function on $[x_0(r), \hat{x}(r)]$.

Proof. It is first shown that

$$\forall \underline{x} \in [x_0(r), \hat{x}(r)], \exists \bar{x}_R(\underline{x}) \in (\underline{x}, 1] \text{ such that } R(\underline{x}, \bar{x}_R(\underline{x})) = 0. \quad (13)$$

Suppose that for some $\underline{x} \in [x_0(r), \hat{x}(r)]$, $R(\underline{x}, x) > 0 \forall x \in (\underline{x}, 1]$. Then taking $x = \hat{x}(r)$,

$$\int_{\underline{x}}^{\hat{x}(r)} [v(\hat{x}(r), y) - r] dF(y) > 0.$$

Since $v_2(x, y) > 0 \forall x, y$, this contradicts the definition of \hat{x} . Suppose instead that for some $\underline{x} \in [x_0(r), \hat{x}(r)]$, $R(\underline{x}, x) < 0 \forall x \in (\underline{x}, 1]$. Since $v_1(x, y) > 0$ for all x, y we must then have

$$R(x_0(r), x) < 0 \quad \forall x \in (x_0(r), 1]$$

which contradicts the hypothesis of Theorem 2. The statement in (13) then follows from differentiability of $R(\cdot, \cdot)$ and the intermediate value theorem. The lemma then follows from the implicit function theorem since, as observed above, $R_1(x, \bar{x}) > 0 \forall \bar{x} > x$ such that $R(x, \bar{x}) = 0$, and $R_2(x, \bar{x}) > 0 \forall x < \bar{x}$ when $\bar{x} \geq \hat{x}(r)$. ■

LEMMA 3. $\lim_{x \uparrow \hat{x}(r)} \bar{x}_R(x) = \hat{x}(r)$.

Proof. From (13), for all $\epsilon \in (0, \hat{x}(r) - x_0(r)]$ there exists $\bar{x}_R(\hat{x}(r) - \epsilon)$ such that

$$\int_{\hat{x}(r) - \epsilon}^{\bar{x}_R(\hat{x}(r) - \epsilon)} \left[v\left(\bar{x}_R(\hat{x}(r) - \epsilon), y\right) - r \right] dF(y) = 0$$

i.e.,

$$\begin{aligned} & \int_{\hat{x}(r)}^{\bar{x}_R(\hat{x}(r) - \epsilon)} \left[v\left(\bar{x}_R(\hat{x}(r) - \epsilon), y\right) - r \right] dF(y) \\ & + \int_{\hat{x}(r) - \epsilon}^{\hat{x}(r)} \left[v\left(\bar{x}_R(\hat{x}(r) - \epsilon), y\right) - r \right] dF(y) = 0. \end{aligned}$$

Since the second term converges to 0 as $\epsilon \downarrow 0$, we must have

$$\lim_{\epsilon \downarrow 0} \int_{\hat{x}(r)}^{\bar{x}_R(\hat{x}(r)-\epsilon)} \left[v(\bar{x}_R(\hat{x}(r)-\epsilon), y) - r \right] dF(y) = 0. \quad (14)$$

Since $\bar{x}_R(\underline{x}) > \hat{x}(r) \forall \underline{x} < \hat{x}(r)$, the definition of $\hat{x}(r)$ and (14) require that

$$\lim_{\epsilon \downarrow 0} \bar{x}_R(\hat{x}(r) - \epsilon) = \hat{x}(r). \quad \blacksquare$$

LEMMA 4. For all $\bar{x} \in [\hat{x}(r), \bar{x}_R(x_0(r))]$, there exists a unique $\underline{x}_Q(\bar{x}) \in [x_0(r), \bar{x}]$ satisfying $Q(\underline{x}_Q(\bar{x}), \bar{x}) = 0$.

Proof. The fact that $Q_1(\underline{x}, \bar{x}) > 0$ for all $\bar{x} \in [\hat{x}(r), \bar{x}_R(x_0(r))]$ ensures that at most one $\underline{x}_Q(\bar{x})$ exists. Suppose first that for some $\bar{x} \in [\hat{x}(r), \bar{x}_R(x_0(r))]$, $Q(x, \bar{x}) > 0 \forall x \in [x_0(r), \bar{x}]$. Taking $x = x_0(r)$ this implies

$$Q(x_0(r), \bar{x}) = \frac{1}{2} \int_{x_0(r)}^{\bar{x}} [v(x_0(r), y) - r] dF(y) > 0.$$

Since $v_2(x, y) > 0 \forall x, y$, this implies $Q_2(x_0(r), \bar{x}) > 0$. Then, since $\bar{x} \leq \bar{x}_R(x_0(r))$, we have $Q(x_0(r), \bar{x}_R(x_0(r))) > 0$, i.e.,

$$\int_{x_0(r)}^{\bar{x}_R(x_0(r))} [v(x_0(r), y) - r] dF(y) > 0.$$

Since $v_1(x, y) > 0 \forall x, y$, this contradicts the definition of $\bar{x}_R(x_0(r))$. So suppose instead that for some $\bar{x} \in [\hat{x}(r), \bar{x}_R(x_0(r))]$, $Q(x, \bar{x}) < 0 \forall x \in [x_0(r), \bar{x}]$. Take $x = \hat{x}(r)$. From the definition of $\hat{x}(r)$ and the fact that $\hat{x}(r) > x_0(r)$ (Lemma 1) we know that for all $\bar{x} \geq \hat{x}(r)$

$$Q(\hat{x}(r), \bar{x}) = \int_0^{\hat{x}(r)} [w(\hat{x}(r), y) - r] dF(y) + \frac{1}{2} \int_{\hat{x}(r)}^{\bar{x}} [v(\hat{x}(r), y) - r] dF(y) > 0$$

giving a contradiction. The result then follows from the differentiability of $Q(\cdot, \cdot)$ and the intermediate value theorem. \blacksquare

LEMMA 5. $\underline{x}_Q(x) \in (x_0(r), \hat{x}(r)) \forall x \in [\hat{x}(r), \bar{x}_R(x_0(r))]$.

Proof. We know $\hat{x}(r) > x_0(r)$, $w_1(x, y) > 0$, and $v_2(x, y) > 0$. Therefore,

$$Q(\hat{x}(r), \bar{x}) = \int_0^{\hat{x}(r)} [w(\hat{x}(r), y) - r] dF(y) + \frac{1}{2} \int_{\hat{x}(r)}^{\bar{x}} [v(\hat{x}(r), y) - r] dF(y) > 0$$

for all $\bar{x} \in [\hat{x}(r), \bar{x}_R(x_0(r))]$. Therefore, since $Q_1(\underline{x}, \bar{x}) > 0 \forall \underline{x} \in [x_0(r), \hat{x}(r)]$, we must have $\underline{x}_Q(x) < \hat{x}(r) \forall x \in [\hat{x}(r), \bar{x}_R(x_0(r))]$. For all $\underline{x} \leq x_0(r)$ and $x \in [\hat{x}(r), \bar{x}_R(x_0(r))]$,

$$Q(\underline{x}, x) = \int_0^{\underline{x}} [w(\underline{x}, y) - r] dF(y) + \frac{1}{2} \int_{\underline{x}}^x [v(\underline{x}, y) - r] dF(y) < 0.$$

Hence $\underline{x}_Q(x) > x_0(r)$. \blacksquare

Now define $\bar{\bar{x}}_R(\cdot)$ as the extension of $\bar{x}_R(\cdot)$ to the domain $[x_0(r), \hat{x}(r)]$ by defining $\bar{\bar{x}}_R(\hat{x}(r)) = \hat{x}(r)$. Lemmas 2–5 show that $\bar{\bar{x}}_R$ is a continuous function from $[x_0(r), \hat{x}(r)]$ onto $[\hat{x}(r), \bar{x}_R(x_0(r))]$ while \underline{x}_Q is a continuous function from $[\hat{x}(r), \bar{x}_R(x_0(r))]$ to $[x_0(r), \hat{x}(r)]$. Define $\Phi(\cdot) : [x_0(r), \hat{x}(r)] \rightarrow [x_0(r), \hat{x}(r)]$ by

$$\Phi(x) \equiv \underline{x}_Q(\bar{\bar{x}}_R(x)).$$

Note that $\Phi(\cdot)$ is continuous since $\bar{\bar{x}}_R(\cdot)$ and $\underline{x}_Q(\cdot)$ are. Brouwer's fixed point theorem then guarantees the existence of a solution to the equation $\Phi(x) = x$, giving a solution to (6') and (9').

To show uniqueness, recall that $Q_1(\underline{x}, \bar{x}) > 0 \forall \bar{x} > \underline{x} \geq x_0(r)$ and that $Q_2(\underline{x}, \bar{x}) < 0$ at any (\underline{x}, \bar{x}) solving (6') and (9'). Hence, $\underline{x}_Q(\cdot)$ is strictly increasing at any solution. Since $\bar{\bar{x}}_R(\cdot)$ is strictly decreasing on $[x_0(r), \hat{x}(r)]$, this implies that at most one solution to (6') and (9') can exist, completing the proof of Theorem 2. ■

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