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Auctions with private uncertainty and resale opportunities

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Abstract

When an auction is followed by an opportunity for resale, bidder valuations are endogenously determined, reflecting anticipated profit from buying/selling in the resale market. These valuations vary with the resale market structure, can differ across auction types, and may be lower or higher than if resale were impossible. Although resale introduces a common value element to the model, revenue equivalence can hold; when it fails, this is due not to affiliation but to differences in information conveyed to the secondary market. Information linkages between markets can also lead to signaling and, in some cases, preclude separation in the auction.

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1. Introduction

Goods sold by auction are often traded later in secondary markets. Bidders for real estate, used cars, artwork, Treasury bills, timber contracts, operating licenses, and antiques, for example, are all likely to consider the option of participating in a resale market when choosing their bids. These options will be particularly important when bidders are uncertain of the values they place on the objects for sale. In such cases, the resale market benefits both winners and losers, since a bidder is neither

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stuck with an object he wins nor stuck without one if he loses. This paper studies auctions at which bidders have noisy signals of their private values but anticipate an improvement in their information and an opportunity for subsequent resale trade. In many (perhaps most) auctions, bidders have noisy estimates of the value the object for sale will provide them: bidders at wholesale used car auctions may be uncertain of demand in their home markets; bidders for a house may be uncertain of future career prospects in the area; bidders for a spectrum license may face idiosyncratic technological uncertainty or be uncertain which complementary licenses they will obtain in later auctions. However, bidders' private uncertainty will be at least partially resolved over time. Hence, while a standard auction may allocate an object to the bidder who expects to value it most highly, there may be gains to trade once new information is available.¹

This paper considers a two-stage model in which an auction in the first stage is followed by a resale auction, held by the first-stage winner. For the first stage I focus on three auctions that are common in practice and have dominated the auction literature: first- and second-price sealed bid auctions and the English auction. I consider two possible resale mechanisms: an optimal auction and an English auction with the option to reject all bids. Examining different first-stage auctions is important because some of the usual similarities between standard auctions carry over while others do not. Focusing on these auctions also enables evaluation of the robustness of prior results to the existence of resale opportunities. Key results also depend on (a) the expected division of surplus in the secondary market and (b) the effects information revealed in the primary market has on resale outcomes. The two resale mechanisms represent opposite extremes in these dimensions, leading to results that span those arising under a wide range of resale market structures. These mechanisms also have appeal of their own. An optimal auction is a natural choice, although it requires commitment by the resale seller. An auction with the option to reject all bids is a selling mechanism observed frequently in practice and is natural when a seller can commit only not to make an unprofitable trade. However, these resale mechanisms serve primarily to illustrate the dependence of equilibrium behavior in the primary market on the structure of the secondary market. Because changes in the resale market structure are captured by changes in the reduced form payoffs to auction winners and losers, it will be clear how the analysis could be extended to any well specified resale market structure.

While the prevalence and potential importance of resale has been recognized at least since Milgrom and Weber [28], it has received little formal attention. This may be due in part to a conjecture that resale offers a motivation for standard common values or affiliated values models.² In the model studied here resale does introduce a common value element, since opponents' signals are correlated with the profit a

¹In [4,12], gains to resale trade arise due to the presence of new buyers in the secondary market. In [11] asymmetries between bidders can result in inefficient allocations, motivating resale trade. Milgrom [27], Kamien et al. [17], and Gale et al. [8] consider complete information models of auctions with resale.

²Comments reflecting this conventional wisdom can be found, for example, in [1,5,6,16,20–23,26,27,32,33].

bidder could make by selling or buying in the resale market. However, a resale opportunity has two fundamental consequences for the first-stage auction that render existing models inadequate.

The first is the endogenous determination of bidder valuations. The option for a winner to sell in the secondary market increases the gain from winning the resale seller effect. However, the option for losing bidders to buy in the secondary market raises the payoff to a loser the resale buyer effect. The value a bidder puts on winning the auction depends on these opposing option values and can be higher or lower than without the resale opportunity. The second consequence is an information linkage created between primary and secondary markets. Because the division of resale surplus can depend on players' beliefs about their opponents, a player's resale profit (and, therefore, the value he puts on winning) can depend on the bid he makes in the primary market due to the information it reveals. Depending on the resale market structure, a bidder can have an incentive to raise his bid in the primary market to signal a high type or lower his bid to signal a low type.³

These phenomena have important implications for the design of markets. If there are no signaling incentives (for example, if there is no informational asymmetry in the resale market), revenue equivalence holds in the primary market despite the common value element introduced by resale. However, the effect of resale on the initial seller's revenue is ambiguous, depending on the relative sizes of the resale seller and resale buyer effects. Revenue equivalence between first- and second-price sealed bid auctions holds even when signaling incentives are present; however, significant differences between sealed bid and English auctions then arise. When an optimal auction is held in the resale market, the resulting signaling incentives preclude existence of a symmetric separating (i.e., efficient) equilibrium in an initial English auction. When an English auction is held in the resale market, revenue equivalence between first-stage English and sealed bid auctions fails. However, this is not due to the "linkage principle" that drives Milgrom and Weber's [28] revenue ranking of standard auctions. Rather, it arises from differences in the information conveyed to the secondary market by English and sealed bid auctions. Finally, these results have important implications for empirical studies of auctions when resale opportunities exist: even when there are no signaling incentives, the mapping between primitives and observables (and therefore the identification of the former from the latter) depends on fine details of the resale market structure.

The following section sets up the model. Section 3 focuses on the special case in which resale takes place under complete information, eliminating the information linkage between the first and second stages. This simplifies the presentation and facilitates focusing on the effects of endogenously determined valuations. Section 4 then addresses the general case in which players have asymmetric information in the secondary market. Section 5 concludes.

³Signaling also arises in the models of Bikhchandani and Huang [4] and Haile [12], although only in the direction leading to higher bids.

2. The model

Consider a two-stage game played by $n \geq 2$ risk neutral buyers who are symmetric ex ante. In the first stage buyers compete for a single indivisible object at an auction of type $a \in \{f, s, e\}$ (first-, second-price, English). Ties are resolved by uniform randomization. If no nonnegative bid is submitted, the game ends and the seller keeps the object. Otherwise, she awards it to the high bidder and announces all bids.⁴ In the second stage the first-stage winner holds another auction, where bidders are now the $n - 1$ first-stage losers. Holding the resale auction is costless. Since I will focus on weakly undominated second-stage strategies, I assume, without loss of generality, that the first-stage winner always holds the resale auction. I consider two specifications of the second-stage auction, each of which is discussed in detail below:

(OA) An *optimal auction*.

(EA) An *English auction* in which no reserve price is announced but the seller is free to reject all bids.

All value from the object itself is received at the end of the second stage, and players do not discount monetary payments between stages. Hence, a bidder who wins the initial auction but sells in the resale market has a payoff equal to the difference between the second-stage and first-stage prices. A player's *use value* is defined as the value he places on the object ignoring any resale opportunities. This terminology is necessary because a bidder's *valuation*—the value he places on winning the auction—will be determined endogenously, taking account of the resale opportunity.

Each bidder i 's use value is denoted by $U_i \in [0, 1]$ and is independent of all other use values.⁵ In stage one, i observes only a private signal $X_i \in [0, 1]$ that is strictly affiliated with U_i but independent of X_j and U_j for all $j \neq i$.⁶ After the first-stage auction, each player learns the realization of his use value.⁷

We can view each X_i as independently and identically distributed according to a distribution $F(\cdot)$, which I assume is strictly increasing on $[0, 1]$ with continuous derivative $f(\cdot)$. Each U_i has conditional distribution $G(\cdot|X_i)$ with support $[0, 1]$.⁸ I

⁴Such bid revelation is required in most government auctions. With complete information in the resale market this assumption is irrelevant. With asymmetric information in the resale market, relaxing this assumption significantly complicates the derivation of continuation payoffs. The seller's optimal announcement policy will depend on the effect of the announced information on surplus extraction in the resale market and on the incentives to signal through bids. This is an issue I will not address here.

⁵Throughout the paper, upper case letters denote random variables and lower case letters their realizations.

⁶See [28] for a discussion of affiliation, which is equivalent here to the multivariate likelihood ratio order (e.g., [34]). Two random variables C and D with joint density $\xi(\cdot, \cdot)$ are *strictly affiliated* if for all $\hat{c} > c$ and $\hat{d} > d$, $\xi(\hat{c}, \hat{d})\xi(c, d) > \xi(\hat{c}, d)\xi(c, \hat{d})$.

⁷Assuming each player's uncertainty were only partially resolved between stages would be equivalent: U_i would then be i 's expectation of his use value conditional on the stage-2 information.

⁸The assumption that the support of $G(\cdot|X_i)$ does not depend on X_i can be relaxed, although this complicates the exposition and requires an appropriate relaxation of the strict affiliation assumption. Several examples below will exploit this relaxation.

assume $G(u|x)$ and the corresponding density $g(u|x)$ are differentiable with respect to both arguments. The structure of the game, the number of bidders, the resale market mechanism, and the distributions $F(\cdot)$ and $G(\cdot|\cdot)$ are common knowledge. Standard arguments (following those in [24,25,28] or [34, Chapter 1]) show that strict affiliation of X_i and U_i implies the following:

Lemma 1. For all $\{x, u\} \in (0, 1)^2$ (i) $G(u|x)$ strictly decreases in x ; (ii) $\frac{1-G(u|x)}{g(u|x)}$ strictly increases in x ; and (iii) given any strictly increasing function ϕ and any $0 \leq a < b \leq 1$, $E[\phi(U_i)|U_i \in [a, b], X_i = x]$ strictly increases in x whenever the expectation exists.

Taking the perspective of a representative bidder i , let Y_j denote the j th highest signal among his opponents and define $\mathbf{Y} \equiv \{Y_1, Y_2, \dots, Y_{n-1}\}$ and $\mathbf{Y}_{-1} \equiv \{Y_2, \dots, Y_{n-1}\}$. Let $\mathbf{Z} \equiv \{Z_1, \dots, Z_{n-1}\}$ denote the order statistics (ordered largest to smallest) of $\{U_j\}_{j \neq i}$ and let $G_k(\cdot|\zeta)$, with derivative $g_k(\cdot|\zeta)$, denote the distribution of Z_k conditional on information ζ .

Lemma 2. For all $z \in (0, 1)$, $G_1(z|\mathbf{y})$ and $G_2(z|\mathbf{y})$ are strictly decreasing in \mathbf{y} .

Proof. See Appendix A.

3. Complete information in the resale market

Consider the case in which use values are common knowledge among all n players in the second stage. I focus on perfect Bayesian equilibria of the two-stage game in symmetric strictly increasing first-stage bidding strategies and weakly undominated second-stage strategies. For simplicity, I will refer such an equilibrium as a *symmetric separating equilibrium*. I begin by deriving expectations of payoffs in the second-stage continuation game under OA and EA.

3.1. Expected second-stage payoffs

The optimal resale auction (OA) consists of the second-stage seller making an ultimatum offer to the opponent with the highest use value. When gains to trade exist, the offer will equal this use value and will be accepted in equilibrium, giving all surplus to the resale seller. Consequently, a bidder who wins the first-stage auction obtains a payoff equal to the highest use value among all players, minus the price paid to the initial seller. Conditional on the signals X_i and \mathbf{Y} , bidder i 's expectation of the maximum use value among all players is given by $W_o(X_i, Y_1, \dots, Y_{n-1})$, where

$$W_o(x, y_1, \dots, y_{n-1}) = \int_0^1 \left[u + \int_u^1 (z_1 - u) dG_1(z_1|\mathbf{y}) \right] dG(u|x).$$

The expected payoff to a losing bidder, $L_o(X_i, Y_1, \dots, Y_{n-1})$, is zero since even if he buys in the resale market he will pay a price equal to his use value.

Now consider the alternative case in which an English auction (EA) is held in the second stage. I follow the convention of modeling an English auction as a “button auction,” in which each bidder depresses a button as the price rises exogenously and releases his button observably to irreversibly exit [28]. While there are many equilibria of the button auction, here all equilibria in weakly undominated strategies have the same outcome as that when each bidder follows the strategy of remaining in the auction until either (i) the price reaches his own use value, or (ii) all other bidders have exited and the price is at least as high as the second-stage seller’s use value. This dependence on the resale seller’s use value results from the fact that the high bid in the resale market can be rejected.⁹

When gains to trade exist, the object is resold and the buyer pays a price equal to the maximum use value among his opponents. So when bidder i wins the first-stage auction, his expected payoff conditional on all the first-stage signals is given by $W_e(X_i, Y_1, \dots, Y_{n-1}) - P$, where P is the price he pays the initial seller and

$$W_e(x, y_1, \dots, y_{n-1}) = \int_0^1 \left[u + \int_u^1 (z_2 - u) dG_2(z_2|\mathbf{y}) \right] dG(u|x).$$

A losing bidder obtains a strictly positive expected payoff equal to the expected maximum of zero and the difference between his own use value and the highest use value among all his opponents:

$$L_e(x, y_1, \dots, y_{n-1}) = \int_0^1 \int_0^u (u - z_1) dG_1(z_1|\mathbf{y}) dG(u|x).$$

3.2. First-stage bidding

3.2.1. Sealed bid auctions

When players choose bids for a sealed bid auction, each has observed only his own signal. As usual, however, in equilibrium the outcome of the auction will itself be informative, revealing to each bidder i either $Y_1 > X_i$ or $Y_1 < X_i$. Bidders must condition on this information when determining their willingness to pay because opponents’ signals are correlated with their use values, which affect all payoffs in the resale market. Given resale market structure $r \in \{o, e\}$ (OA or EA) define

$$w_r(x, y) = E[W_r(X_i, Y_1, \dots, Y_{n-1})|X_i = x, Y_1 = y]$$

and

$$\ell_r(x, y) = E[L_r(X_i, Y_1, \dots, Y_{n-1})|X_i = x, Y_1 = y].$$

⁹I allow the top resale bidder to let the price rise past that at which his last opponent exits since it will sometimes be optimal for him to do so and it is optimal for the resale seller to allow this.

Now suppose $b(\cdot)$ is a symmetric strictly increasing equilibrium bid function for a second-price sealed bid auction when the resale market is of type r . The expected payoff to a bidder with signal x who bids $b(\tilde{x})$ is¹⁰

$$\pi(x, \tilde{x}) \equiv \int_0^{\tilde{x}} [w_r(x, y) - b(y)] dF_1(y) + \int_{\tilde{x}}^1 \ell_r(x, y) dF_1(y),$$

where $F_1(x) \equiv F(x)^{n-1}$. Differentiating with respect to \tilde{x} and setting $\tilde{x} = x$ gives the unique candidate equilibrium bid function

$$b_{s,r}(x) = v_r(x, x),$$

where the function

$$v_r(x, y) \equiv w_r(x, y) - \ell_r(x, y) \quad (1)$$

can be interpreted as a bidder's expected *valuation*, i.e., the expected value of winning the auction rather than losing, conditional on the realizations of X_i and Y_1 . Theorem 1 then generalizes the familiar result that in a second-price auction players bid their expected valuations, conditioned on an assumption that the highest signal among their opponents is the same as their own.

Theorem 1. *If a second-price sealed bid auction is held in the first stage, $b_{s,r}$ is the unique symmetric separating equilibrium bidding strategy.*

Proof. See Appendix A.

Eq. (1) illustrates the two opposing effects of the resale opportunity on bidders' expected valuations. The fact that $w_r(x, y) \geq E[U_i | X_i = x]$ is immediate from voluntary participation in the resale market and holds with a strict inequality under both OA and (for $n > 2$) EA. This demonstrates the *resale seller effect*: the first-stage winner gains the option value of selling in the resale market. However, $\ell_r(x, y)$ is also nonnegative under any resale market structure satisfying voluntary participation. The fact that $\ell_r(x, y)$ is subtracted from $w_r(x, y)$ to obtain $v_r(x, y)$ reflects the *resale buyer effect*: because the auction need not determine the final allocation, the value of winning the auction is limited (reduced) by the option value of buying in the secondary market.

The analysis is similar for a first-price sealed bid auction. Indeed, if one treats

$$V_r(X_i) \equiv v_r(X_i, X_i) \quad (2)$$

as bidder i 's private value, a similar optimization problem leads to the equilibrium bid function

$$b_{f,r}(x) = E[V_r(Y_1) | Y_1 \leq x]$$

¹⁰ Because there is probability zero of a tie in equilibrium, any bid outside the range of $b(\cdot)$ is equivalent to some bid in this range. Similar arguments (see [13]) apply for the other auction types below.

i.e., the equilibrium bid function for a first-price sealed bid auction without resale in which bidders had i.i.d. private values defined by (2).

Theorem 2. *If a first-price sealed bid auction is held in the first stage, $b_{f,r}$ is the unique differentiable symmetric separating equilibrium bid function.*

Proof. See Appendix A.

3.2.2. English auctions

Following Milgrom and Weber [28] I model an English auction in a sequence of phases, beginning in phase 0 and entering a new phase each time a bidder drops out. These exits are observable and irreversible. A bidding strategy consists of a sequence of bid functions, one for each phase, mapping a player’s signal and the prior exit prices into a price, giving the supremum of the set of prices at which he will remain in the auction if no opponent exits first.¹¹ In equilibrium, bids strictly increase in signals, so conditioning on the exits observed prior to phase k is equivalent to conditioning on

$$\mathbf{Y}_{-1}^k \equiv \begin{cases} \{Y_{n-k}, \dots, Y_{n-1}\}, & k = 1, \dots, n - 2, \\ \emptyset & k = 0. \end{cases}$$

In addition, the outcome of phase k will reveal something about the realization of Y_{n-k-1} , about which each bidder has a prior in phase k given by

$$\hat{F}_k(s|\mathbf{y}_{-1}^k) = \begin{cases} 1 - \left(\frac{1-F(s)}{1-F(y_{n-k})}\right)^{n-k-1}, & k > 0, \\ 1 - (1 - F(s))^{n-1}, & k = 0. \end{cases}$$

For phases $k = 0, \dots, n - 2$ (those in which at least two bidders remain) let

$$\hat{w}_r^k(x, y, \mathbf{y}_{-1}^k) = E[W_r(X_i, Y_1, \dots, Y_{n-1})|X_i = x, Y_{n-1-k} = y, \mathbf{Y}_{-1}^k = \mathbf{y}_{-1}^k]$$

and

$$\hat{\ell}_r^k(x, y, \mathbf{y}_{-1}^k) = E[L_r(X_i, Y_1, \dots, Y_{n-1})|X_i = x, Y_{n-1-k} = y, \mathbf{Y}_{-1}^k = \mathbf{y}_{-1}^k].$$

These expectations are the analogs of those derived above for the sealed bid auctions, accounting for the additional information revealed prior to phase k of an English auction. Let

$$\hat{v}_{e,r}^k(X_i, Y, \mathbf{Y}_{-1}^k) = \hat{w}_r^k(X_i, Y, \mathbf{Y}_{-1}^k) - \hat{\ell}_r^k(X_i, Y, \mathbf{Y}_{-1}^k).$$

Theorem 4 shows that, analogous to the equilibrium characterized by Milgrom and Weber [28] for an affiliated values English auction without resale, there is an equilibrium in which in each phase k , each bidder i uses the bid function

$$b_{e,r}^k(X_i; \mathbf{Y}_{-1}^k) = E[\hat{v}_{e,r}^{n-2}(X_i, Y_1, \mathbf{Y}_{-1})|X_i, Y_1 = \dots = Y_{n-1-k} = X_i, \mathbf{Y}_{-1}^k]$$

¹¹Off the equilibrium path a bidding strategy may specify a phase- k bid below the phase- $(k - 1)$ exit price. A player follows such a strategy by exiting immediately when phase k begins.

i.e., each bids his expected valuation, conditional on an assumption that all remaining opponents have the same type he does. Theorem 3 first shows that this is the only possible equilibrium strategy for phase $n - 2$. We can consider this the last phase, since the last remaining bidder optimally exits immediately when his final opponent does, regardless of his type. Hence, the term “separating” will be used to refer to an equilibrium in which there is separation in phases $0, \dots, n - 2$.

Theorem 3. *If an English auction is held in the first stage, in any symmetric separating equilibrium, bidding in phase $n - 2$ follows the bid function $b_{e,r}^{n-2}(x; \mathbf{y}_{-1}) = \hat{v}_r^{n-2}(x, x, \mathbf{y}_{-1})$.*

Proof. See Appendix A.

Theorem 4. *If an English auction is held in the first stage, there exists a symmetric separating equilibrium in which bidders use the bidding strategy $b_{e,r}^k(x; \mathbf{y}_{-1}^k)$ in each phase $k = 0, \dots, n - 2$.*

Proof. See Appendix A.

3.3. Resale and common values

Conventional wisdom suggests that a resale opportunity creates common values. Common values exist whenever a bidder’s opponents have private information about the value he would obtain from acquiring ownership of the object at the auction.

Definition. In any auction, let V_i denote the gain in utility bidder i receives by obtaining the object at the auction (gross of the price he pays). Let X_i denote i ’s private information and X_{-i} that of his opponent(s). Bidders have *common values* if for all i , $E[V_i|X_i] \neq E[V_i|X_i, X_{-i}]$.

In standard models, where V_i is just i ’s use value, this definition rules out pure private values and any environment in which bidders have no private information. All other affiliated values structures [28] satisfy the definition, as do hybrid models with multi-dimensional types like that in [35]. In the present model, V_i is the difference between the payoff to i when he wins the auction and that when he loses. As the analysis above illustrates, this payoff depends on the realization of U_{-i} , implying that its expectation depends on X_{-i} .¹²

Remark 1. Bidders in the first-stage auction have common values.

¹²Note that the definition of common values is not limited to the special case of *pure* common values, where every bidder has the same unknown value for the object. However, under OA the value of winning the object is $V_i = V = \max_j U_j$ for every bidder i , which is an example of pure common values.

In a sealed bid auction, for example,

$$E[V_i|X_1, \dots, X_n] = W(X_i, Y_1, \mathbf{Y}_{-1}) - L(X_i, Y_1, \mathbf{Y}_{-1})$$

which varies with Y_1 and \mathbf{Y}_{-1} . Due to the anticipated interaction in the resale market, bidders' first-stage signals play a role similar to that of signals of a common value component in standard affiliated values models: the higher are opponents' signals, the higher is the expected (gross) payoff from winning and the lower is the expected payoff from losing. Hence, bidders are willing to pay more to win when they believe their opponents' signals are high.

Not surprisingly, then, there is a winner's curse calculation implicit in the bidding strategies above. This is not to say that bidders are willing to pay less as a result of the resale market—we will see below that this may or may not be the case. However, each bidder realizes that winning the auction would imply that all other signals are lower than his own. This information reduces the expected value of the option to sell in the secondary market while raising the value of the option to buy. Both effects reduce the value of winning the auction. There is a parallel *loser's curse*: losing the auction reveals that others are likely to have high use values, implying a lower expected payoff from attempting to buy in the resale market than the unconditional expectation would indicate and a higher expected payoff from selling in the resale market if he were to win instead.¹³

It is important to emphasize that Remark 1 does *not* imply that models of common value auctions without resale accurately describe auctions with resale. Haile [15], for example, has shown that in this type of model each bidder's willingness to pay increases with the number of competitors—a prediction that is inconsistent with both private and common values models without resale.¹⁴ Below we will see other differences with implications for efficiency, seller revenue, and empirical analysis of bids.

3.4. Seller revenues

3.4.1. Revenue equivalence

Revenue equivalence of the three first-stage auctions is now easily verified. This contrasts with what one might expect based on well-known results showing revenue superiority of English auctions in affiliated values auctions [28] and the observation that bidders have common values. However, while resale introduces a common value component to the auction, it does not introduce correlation of bidders' private information. Consequently the “linkage principle” ([27,28]) has no bite here.

¹³Pesendorfer and Swinkels [30] and Feddersen and Pesendorfer [7] discuss the loser's curse in other environments.

¹⁴This holds because, as in the present model, the number of bidders is a signal of the level of competition among buyers in the resale market.

Theorem 5. *Holding the rules of the resale market fixed, the first-stage seller's expected revenues are identical in the first-price, second-price, and English auctions.*

Proof. See Appendix A.

3.4.2. Does the initial seller benefit from the resale market?

The presence of both the resale seller effect and resale buyer effect implies that from the initial seller's perspective the desirability of a resale market is ambiguous. If the resale seller extracts a sufficient share of the resale surplus, the resale seller effect will dominate and the existence of the resale market will enhance the initial seller's revenue.¹⁵ Under OA, for example, all the bargaining power is given to the resale seller and, as the following theorem shows, the initial seller prefers that a resale opportunity exist. Under EA, however, the resale seller extracts less surplus (for finite n). When n is small, the resale buyer effect dominates and the seller would like to ban resale.

Theorem 6. *Under OA the initial seller's expected revenue is higher than without a resale market. Under EA, there exist integers \bar{N} and \underline{N} , with $\bar{N} \geq \underline{N} \geq 2$, such that existence of the resale market lowers the initial seller's expected revenue when $n \leq \underline{N}$ and raises it when $n > \bar{N}$.*

Proof. See Appendix A.

In a government auction, the seller often has the ability to ban resale trade (in the case of procurement auctions, it often does). When banning resale is infeasible, the seller may be able to influence the precision of bidders' estimates of their use values, either through revelation of information relevant to these assessments or through the timing of the sale. If bidders know their use values (e.g., if the seller can wait until bidders' uncertainty has been resolved to hold the auction), the resale market is effectively eliminated.¹⁶ As the following example illustrates, when the resale seller effect is weak the initial seller may prefer that bidders know their use values. However, when the resale seller effect is strong, the initial seller may prefer that bidders have noisy signals so that the high expected surplus extraction in the resale

¹⁵Kamien and Samet [17] consider a two-bidder complete information model of auctions with subcontracting. They do not explicitly consider seller revenues but derive results that imply dependence of revenues on whether it is the winner or loser who makes an ultimatum subcontracting offer.

¹⁶Note, however, that the case in which signals are noisy and resale is banned is different from that in which the resale opportunity is made superfluous by eliminating noise in the signals. In the first case bidders have private values that are expectations of their use values. In the second case, the private values are the use values themselves. Since the distribution of U_i is typically different from that of $E[U_i|X_i]$, the seller's expected revenues (the expectation of the second highest private value) will generally be different in these two cases.

market can be passed through in the first-stage bids.¹⁷ In such cases practices of limiting pre-auction inspections as done at used car auctions [9], or holding auctions before all idiosyncratic uncertainty is resolved as done at Forest Service timber auctions [15] would enhance seller revenues.

Example 1. Let each U_i be distributed uniformly on $[0, 1]$ while each X_i is uniform on $[U_i, U_i + t]$, where $t \geq 0$ is common knowledge. Let $U_{j:n}$ denote the j th highest use value. When there is no resale opportunity, each bidder i bids $E[U_i|X_i]$. If $t = 0$, this gives the seller expected revenue $E[U_{2:n}] = \frac{n-1}{n+1}$. As $t \rightarrow \infty$, almost all signals become pure noise,¹⁸ causing bidders to submit bids approaching the unconditional expectation $E[U_i] = \frac{1}{2}$, giving revenues converging to $\frac{1}{2}$. For $n = 3$, this limit is the same as the revenue with $t = 0$; however, for $n > 3$, $\frac{1}{2} < \frac{n-1}{n+1}$. Fig. 1 illustrates the seller’s expected revenue as a function of t for $n = 2, 3, 4$. For all $n > 3$, the seller prefers that signals be as precise as possible.

The seller’s preference can be very different with a resale opportunity. Under OA the first-stage winner obtains payoff $\max_i u_i$, while a first-stage loser gets nothing. Hence, each bidder i bids

$$E[\max_j U_j | X_i, X_k \leq X_i \ \forall k \neq i]. \tag{3}$$

When $t = 0$, the conditioning in (3) implies that there will be no resale trade, so each bidder i bids $U_i = X_i$ and the seller obtains expected revenue $E[U_{2:n}] = \frac{n-1}{n+1}$ as above. As $t \rightarrow \infty$, this winner’s curse vanishes and all bids converge to $E[U_{1:n}]$. Hence, regardless of n , the seller prefers that signals be as noisy as possible. Fig. 2 illustrates for the case $n = 2$.

Under EA, however, the resale seller extracts less surplus—particularly for small n . The case $t = 0$ is as above. When $t \rightarrow \infty$,

$$w_e(x, y) \rightarrow E[\max\{U_i, \text{second highest of } U_j \ j \neq i\}] = \frac{1 - n + n^2}{n(n + 1)},$$

$$\ell_e(x, y) \rightarrow Pr\left(U_i > \max_{j \neq i} U_j\right) E\left[U_i - \max_{j \neq i} U_j | U_i > \max_{j \neq i} U_j\right] = \frac{1}{n(n + 1)}.$$

This gives $\lim_{t \rightarrow \infty} b(x) = \frac{n-1}{n+1} \forall x$. Fig. 3 illustrates for the case $n = 2$, where $t = 0$ is optimal.

¹⁷Similar intuition underlies a result in [18]. In their model, buyers in the primary market have no private information, as when $t \rightarrow \infty$ in the example below. They assume a continuum of buyers and market clearing prices in both markets. Allowing variation in the quantity sold, they show that a sufficiently tight restriction on quantity ensures that the expected marginal valuation lies above the average valuation, causing a seller to prefer existence of a resale market.

¹⁸A signal of $X_i = 0.01$ is still very informative, as is $X_i = t + 0.99$; however, as $t \rightarrow \infty$ informative signals (those outside the interval $[1, t]$) are drawn with probability approaching zero.

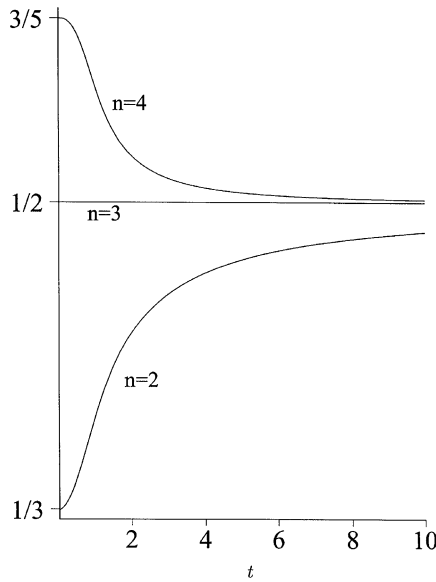


Fig. 1. Expected revenues (no resale).

4. Asymmetric information in the resale market

Now suppose bidders’ information in the second stage consists only of their own signals, their own use values, and any information revealed by the first-stage bids. Now a bidder’s payoff in the resale market may depend on his first-stage bid, due to the information it reveals to his opponents. In particular, in equilibrium each player i ’s first-stage bid reveals his type X_i and, therefore, the conditional distribution $G(\cdot|X_i)$ of his use value.

In the optimal resale auction (OA), the resale seller will use these inferred distributions to construct the optimal selling mechanism as in [29].¹⁹ I assume Myerson’s regularity condition which, given the differentiability assumed above, can be written as

$$\forall x, \quad \frac{\partial}{\partial u} \left[u - \frac{1 - G(u|x)}{g(u|x)} \right] > 0 \quad \forall u \in (0, 1). \tag{4}$$

Because the information revealed by the first-stage bids implies asymmetric distributions for resale buyers’ use values, bidders whose losing first-stage bids reveal

¹⁹ Unlike buyers in Myerson’s model, resale buyers will not know the resale seller’s use value. Since use values are private and preferences are quasi-linear, this is inconsequential in the case of discrete types [19]. One may conjecture that the same is true in the case of continuous types considered here. However, strictly speaking, the optimal auction considered here should be interpreted as an optimal selling mechanism from a restricted class, such as that considered in [36].

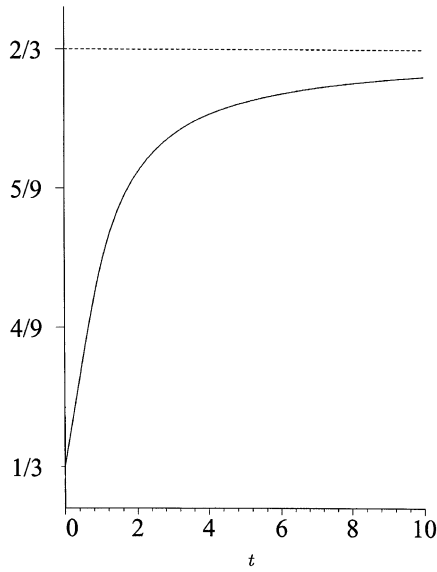


Fig. 2. Expected revenues (OA).

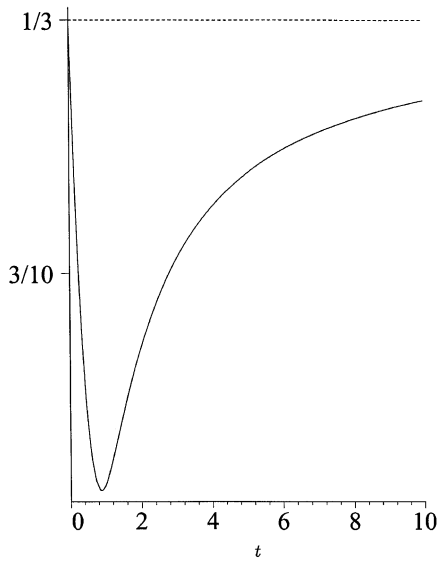


Fig. 3. Expected revenues (EA).

high signals will be discriminated against in the resale auction. Since higher bids imply higher signals in equilibrium, a losing bidder's expected payoff decreases in his first-stage bid. In contrast, because beliefs about the resale seller's use value have no

effect on resale buyer strategies, the first-stage winner’s payoff in the resale market does not depend on his first-stage bid.

The situation is quite different under EA. There the highest bidder in the resale auction must choose a final bid (ultimatum offer) based on his beliefs about the seller’s type, which are determined by the information revealed by the first-stage bids. Following a sealed-bid auction, for example, he will have inferred the resale seller’s signal from his first-stage bid. Higher signals imply that a higher use value is more likely. Since a buyer optimally offers a higher price when he believes the seller is likely to place a high value on the object, the resale seller receives a more favorable offer when his first-stage bid was high; i.e., his expected payoff in the resale market increases in his first-stage bid. Because beliefs about buyer’s use values are irrelevant under EA, the payoff of a losing bidder does not depend on his first-stage bid. The situation is similar following a first-stage English auction although, as we will see below, only a lower bound on the signal of the first-stage winner will be inferred from bids, implying that marginal changes in the winner’s own bid do not affect his resale profit.

The analysis of the second stage under EA is simplified by the following regularity assumptions, which ensure a unique optimal ultimatum offer:²⁰

$$\forall x, \quad \frac{\partial}{\partial u} \left[u + \frac{G(u|x)}{g(u|x)} \right] > 0 \quad \forall u \in (0, 1), \tag{5}$$

$$\forall y, \quad \frac{\partial}{\partial u} \left[u + \frac{\int_y^1 G(u|x) dF(x)}{\int_y^1 g(u|x) dF(x)} \right] > 0 \quad \forall u \in (0, 1). \tag{6}$$

Appendix B provides a complete analysis of each of these resale continuation games. The effects of the information linkages just described are accounted for by adding the argument \tilde{x} to the expected payoff functions used previously, giving

$$w_r(x, y, \tilde{x}), \quad \ell_r(x, y, \tilde{x}), \quad v_r(x, y, \tilde{x})$$

and

$$\hat{w}_r^k(x, y, \mathbf{y}_{-1}^k, \tilde{x}), \quad \hat{\ell}_r^k(x, y, \mathbf{y}_{-1}^k, \tilde{x}), \quad \hat{v}_r^k(x, y, \mathbf{y}_{-1}^k, \tilde{x}).$$

In each case, the expectations are now conditioned on the additional information that the bidder’s own first-stage bid (for phase k in the case of an English auction) was the equilibrium bid of a bidder with signal \tilde{x} . Appendix B derives explicit expressions for these expectations. As in the preceding analysis, only a few key features of these expectations are needed to characterize equilibrium bidding in the first stage.

²⁰Although (6) is nonstandard, by defining $\tilde{G}(u|x) \equiv \int_x^1 G(u|t) dF(t)$ it should be clear that this condition has the same interpretation as (5).

4.1. First-stage sealed bid auction

Consider first a second-price auction. A bidder with signal x chooses \tilde{x} to maximize

$$\pi(x, \tilde{x}) \equiv \int_0^{\tilde{x}} [w_r(x, y, \tilde{x}) - b(y)] dF_1(y) + \int_{\tilde{x}}^1 \ell_r(x, y, \tilde{x}) dF_1(y).$$

The first-order condition gives the unique candidate equilibrium bid function

$$\tilde{b}_{s,r}(x) = v_r(x, x, x) + \psi_r(x) + \lambda_r(x),$$

where

$$\psi_r(x) \equiv \int_0^x D_3[w_r(x, y, x)] \frac{f_1(y)}{f_1(x)} dy,$$

$$\lambda_r(x) \equiv \int_x^1 D_3[\ell_r(x, y, x)] \frac{f_1(y)}{f_1(x)} dy$$

and $D_3[\cdot]$ represents the partial derivative with respect to the third argument of the function inside the brackets.²¹ This proves the following result.

Theorem 7. *If a second-price sealed bid auction is held in the first stage, in any symmetric separating equilibrium players bid according to $\tilde{b}_{s,r}$.*

The key difference from the bid function in Theorem 1 is the presence of the terms $\psi_r(x)$ and $\lambda_r(x)$, which reflect responses to incentives to signal through bids. As shown in Appendix B, the resale seller may benefit when resale buyers believe his own use value is high, implying $D_3[w_r(x, y, \tilde{x})] \geq 0$. This inequality is strict under EA, so in this case the signaling incentives push bids above bidders' expected valuations. Similarly, because a resale buyer may benefit from having opponents (including the resale seller) believe he has a low type, $\lambda_r(x) \leq 0$. This inequality is strict under OA, so that signaling leads to bids below expected valuations.

Theorem 7 does not address existence of a symmetric separating equilibrium. As in many dynamic models with asymmetric information, useful sufficient conditions on primitives that ensure existence are difficult to identify. The following example considers a simple but natural information structure that suggests why this is particularly difficult in this model. In this example, one can verify numerically that $\tilde{b}_{s,r}(\cdot)$ is a strictly increasing equilibrium bidding strategy under both OA and EA. However, due to the signaling incentives, a bidder's objective function is not quasi-concave in his own bid given equilibrium bidding by his opponents.

²¹The relevant derivatives are given in Appendix B.

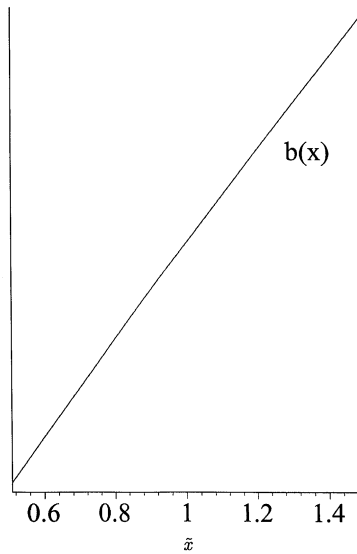


Fig. 4. Equilibrium bid function, second-price auction.

Example 2. Suppose $n = 2$ and each X_i has a uniform distribution on $[1/2, 3/2]$,²² with U_i uniform on $[X_i - \frac{1}{2}, X_i + \frac{1}{2}]$. Fig. 4 illustrates the strictly increasing bid function $\tilde{b}_{s,e}(x) = v_e(x, x, x) + \int_{\frac{1}{2}}^x D_3[w_e(x, y, x)] dy$ prescribed by Theorem 7. Fig. 5 shows the objective function $\pi(x, \tilde{x})$ for a bidder with type $x = 0.8$. The shape of this function is typical of those obtained for all types: standard sufficient conditions (like quasiconcavity in \tilde{x}) that would guarantee that the first-order condition characterizes an optimum fail. Nonetheless, $\pi(x, \tilde{x})$ is always maximized at $\tilde{x} = x$. The same is true under OA, where the failure of quasiconcavity occurs for small rather than large values of \tilde{x} , but $\pi(x, \tilde{x}) \leq \pi(x, x) \forall x, \tilde{x}$.

The analysis is similar for a first-price auction, and a similar necessary condition can be given.

Theorem 8. Let $\theta_r(x) = v_r(x, x, x) + \psi_r(x) + \lambda_r(x)$. If a first-price sealed bid auction is held in the first stage, in any symmetric separating equilibrium in differentiable first-stage strategies, players use the bid function $\tilde{b}_{f,r}(x) = \frac{\int_0^x \theta_r(t) dF_1(t)}{F_1(x)}$.

Proof. See Appendix C.

²²One could obviously define an equivalent signal $\chi_i = X_i - \frac{1}{2}$ that would have support $[0, 1]$.

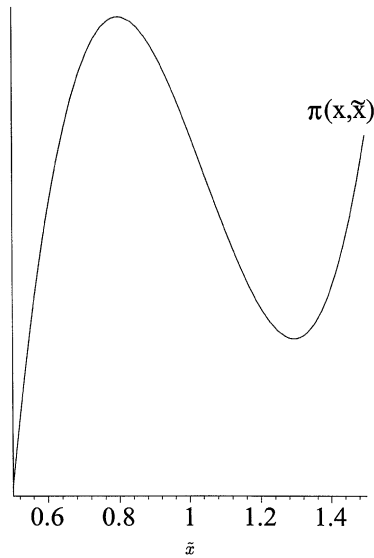


Fig. 5. Objective function for $x = 0.8$.

4.2. First-stage English auction

Unlike a sealed bid auction, the winning bidder’s signal is not inferred from the first-stage bids in an English auction, eliminating the winner’s signaling incentive. Under OA this is immediate: beliefs about the winner’s type do not affect the resale outcome, so a winning bidder who makes a bid above the exit price of his last opponent would only reduce his payoff. Under EA the final bidder would like others to believe he has a high type; however, winning bidders with lower types place higher probability on accepting a given resale price and, therefore, receive a greater benefit from a given improvement in this price. Hence, if there were an equilibrium in which a winner with a given type could induce more favorable beliefs by making a “final bid” once all opponents have exited, this final bid would be even more attractive to a lower type, precluding separation. Therefore, any equilibrium must be one in which all types of bidder pool in phase $n - 1$.

Lemma 3. *In any perfect Bayesian equilibrium in which bidding in phases $0, \dots, n - 2$ follows separating strategies, the exit price of the winning bidder is independent of his type.*

Proof. See Appendix C.

This lack of separation eliminates the signaling incentive seen in the sealed bid auctions under EA. While losers infer a lower bound on the winner’s type based on the price he pays, marginal changes in his phase- $(n - 2)$ bid do not influence

opponents’ second-stage beliefs; i.e.,

$$D_4[\hat{w}_r^{n-2}(x, y, \mathbf{y}_{-1}, \tilde{x})] = 0 \quad \forall x, y, \mathbf{y}_{-1}, \tilde{x}.$$

There is also a more subtle effect under EA: the lack of separation by the first-stage winner of an English auction enables him to extract greater information rents in the resale market than he would following a sealed-bid auction. This shows up as a difference in the expected valuations:

Lemma 4. $E[\hat{v}_e^{n-2}(X_i, X_i, \mathbf{Y}_{-1}, X_i) | X_i = x] > v_e(x, x, x) \quad \forall x \in (0, 1).$

Proof. See Appendix C.

Under EA, the remaining question is at what price pooling occurs in phase $n - 1$. One natural price is that at which the next-to-last bidder drops out. Theorem 9 characterizes the set of equilibria in which this is the pooling price—i.e., when the winning bidder exits immediately once all opponents have exited. While there are infinitely many such equilibria, all yield the same price.²³ As before, the term “separating” here refers to separation in phases $0, \dots, n - 2$ only.

Theorem 9. *Under EA, bid functions $\{\tilde{b}_{e,e}^0, \tilde{b}_{e,e}^1, \dots, \tilde{b}_{e,e}^{n-1}\}$ are symmetric separating equilibrium strategies for each phase $k < n - 1$ of a first-stage English auction if each $\tilde{b}_{e,e}^k$ is strictly increasing and the following conditions hold:*

- (i) $\tilde{b}_{e,e}^{n-1}(x; \{y_1, \mathbf{y}_{-1}\}) = \tilde{b}_{e,e}^{n-2}(y_1; \mathbf{y}_{-1}) = \hat{v}_e^{n-2}(y_1, y_1, \mathbf{y}_{-1}, y_1) \quad \forall x \geq y_1, \mathbf{y}_{-1};$
- (ii) $\tilde{b}_{e,e}^k(x; \mathbf{y}_{-1}^k) \leq E[\hat{v}_e^{n-2}(X_i, Y_1, \mathbf{Y}_{-1}, X_i) | X_i = Y_1 = \dots = Y_{n-k-1} = x, \mathbf{Y}_{-1}^k = \mathbf{y}_{-1}^k] \quad \forall x, \mathbf{y}_{-1}^k;$
- (iii) $\tilde{b}_{e,e}^{k+1}(x; \{x, \mathbf{y}_{-1}^k\}) \geq \tilde{b}_{e,e}^k(x; \mathbf{y}_{-1}^k) \quad \forall x, \mathbf{y}_{-1}^k.$

Proof. See Appendix C.

Suppose, for example, that for all x, \mathbf{y}_{-1}^k and $k < n - 2$

$$\tilde{b}_{e,e}^k(x; \mathbf{y}_{-1}^k) = \beta E[\hat{v}_e^{n-2}(X_i, Y_1, \mathbf{Y}_{-1}, X_i) | X_i = Y_1 = \dots = Y_{n-k-1} = x, \mathbf{Y}_{-1}^k = \mathbf{y}_{-1}^k]$$

while $\tilde{b}_{e,e}^{n-1}(x; \{y_1, \mathbf{y}_{-1}\}) = \tilde{b}_{e,e}^{n-2}(y_1; \mathbf{y}_{-1}) = \hat{v}_e^{n-2}(y_1, y_1, \mathbf{y}_{-1}, y_1) \quad \forall x \geq y_1, \mathbf{y}_{-1}$. For any $\beta \in (0, 1]$, these bid functions satisfy conditions (i)–(iii); the equilibrium with $\beta = 1$ is

²³ An analogous result holds for the English auction without resale (see [3]) and for the English auction in the setting of Section 3. With the restriction to equilibria in which no final bid is made in phase $n - 1$, necessity of conditions (i)–(iii) in Theorem 9 is straightforward: (iii) is required for separation; necessity of the second equality in (i) follows from a first-order condition [15]; then without (ii), (i) could not yield separation in phase $n - 2$.

analogous to Milgrom and Weber's [28] equilibrium for the English auction without resale.

Equilibria in which condition (i) is violated can also be constructed, using forcing beliefs off the equilibrium path to induce a final bid by the winning bidder once his last opponent drops out. Begin with strategies satisfying conditions (ii) (with strict inequality) and (iii) of Theorem 9 for $k < n - 2$. Now suppose the next-to-last bidder, with type y_1 , exits at a price below $\hat{v}_e^{n-2}(y_1, y_1, \mathbf{y}_{-1}, y_1)$ but the winning bidder bids $\hat{v}_e^{n-2}(y_1, y_1, \mathbf{y}_{-1}, y_1)$ in phase $n - 1$. This can yield another equilibrium in which the winning bidder effectively makes the type- y_1 player's bid for him. He will be willing to do this if failure to do so results in sufficiently unfavorable beliefs about his type. Appendix D provides a more detailed discussion of these "final bid equilibria." However, these equilibria still yield a price of $\hat{v}_e^{n-2}(y_1, y_1, \mathbf{y}_{-1}, y_1)$. Hence, up to an indeterminacy regarding which bidder actually makes the price-setting bid of $\hat{v}_e^{n-2}(y_1, y_1, \mathbf{y}_{-1}, y_1)$, Theorem 9 characterizes the full set of symmetric separating equilibria.

Because losing bidders' payoffs depend on their first-stage bids under OA, the results can be starkly different. Instead of a continuum of symmetric separating equilibria, there is none.

Theorem 10. *Under OA, if $n > 2$ and an English auction is held in the first stage, there exists no symmetric separating equilibrium.*

Proof. See Appendix C.

The intuition is straightforward. A marginal change in a player's bid during any phase $k < n - 2$ affects the auction outcome (in particular, whether he ultimately wins or loses or the price he pays if he wins) only in the case of a tie, which occurs with probability zero in a separating equilibrium. However, since $D_4[\hat{v}_0^k(x, y, \mathbf{y}_{-1}^k, \tilde{x})] < 0$, a marginal reduction in his bid in phase $k < n - 2$ is always beneficial: conditional on losing—and a bidder will ultimately lose if a marginal change in his bid in phase $k < n - 2$ matters at all—a lower bid is always preferred. Hence, bidders would always want to deviate from a separating bidding strategy. This is not the case in a two-bidder auction or, equivalently, in phase $n - 2$ of an English auction; in fact, with two bidders, the English and second-price sealed bid auctions are equivalent under OA. There a marginal change in one's bid still has a signaling effect, but it may also change the identity of the winner.²⁴

²⁴If bidders observe opponents' exits with sufficient noise, existence of a symmetric separating equilibrium might be restored, since then bidders might always put positive probability on being in phase $n - 2$. Evaluating this possibility requires a different model of English auctions, which is not pursued here.

4.3. Revenue equivalence?

The signaling incentives discussed above fundamentally alter the form of equilibrium bidding strategies and can preclude the existence of a symmetric separating equilibrium. However, revenue equivalence of the sealed bid auctions still holds whenever a symmetric separating equilibrium exists. This follows immediately from existing results [29,31] and the observation that in a sealed bid auction, equilibrium bidding is exactly as if bidders had independent private values given by²⁵

$$\theta_i = \theta_r(X_i).$$

No general statement can be made regarding revenues from an English auction under OA based on the results above. Under EA, however, revenue equivalence of the English and sealed bid auctions fails. The reasons are most easily seen in a comparison of the English and second-price sealed bid auctions. First, in the sealed bid auction bidders bid their expected valuations plus a positive signaling term $\psi_\epsilon(x)$. This signaling component is absent in a first-stage English auction. Alone, this would suggest lower bids in the English auction. However, there is also an enhancement of the resale seller's information rents when he avoids revealing his first-stage signal to resale buyers. As noted in Lemma 4, this implies that expected valuations themselves are higher in the English auction. This leaves the revenue ordering ambiguous. Whether general sufficient conditions can be given for ranking the sealed bid and English auctions is an open question; however, in Example 2 above, the expected revenue is 0.817 for the second-price auction while that for the English auction is 0.829, showing that revenue equivalence fails.²⁶ The following theorem summarizes these results.

Theorem 11. *Under EA or OA, the first- and second-price sealed bid auctions yield the same expected revenue in a symmetric separating equilibrium. Under EA, revenue equivalence between the sealed bid and English auctions fails.*

Note that because bidders' signals are independent, it is clear that the failure of revenue equivalence under EA is not due to the linkage principle responsible for Milgrom and Weber's [28] revenue rankings of affiliated values auctions without resale. Rather, differences in the information conveyed to the resale market by different first-stage auctions lead to differences in the interactions between bids in the first-stage auction and the division of surplus in the resale market.

²⁵Note that when an equilibrium exists, $\theta_r(\cdot)$ must be strictly increasing. Note also that this equivalence need not hold if the initial seller followed a policy of announcing only the price paid rather than all bids, since this would imply announcing the resale seller's type after a first-price auction but a (possibly unnamed) resale buyer's type after a second-price auction.

²⁶Examples are difficult to solve for expected revenues, requiring multiple integration with respect to complicated conditional distributions and implicitly defined functions as both integrands and limits. The calculations reported above were performed numerically.

5. Conclusion

The model studied here captures elements of strategic behavior likely to arise in many auctions in which bidders have imperfect signals of their use values or their use values change over time. Equilibrium bidding is based on valuations determined both by bidders' own use values and by the equilibrium option values of buying and selling in the secondary market. Because these option values vary with the allocation of rents in the resale market, the initial seller's preference over the existence of a resale opportunity and over the precision of bidders' signals depends on the resale market structure. The endogeneity of valuations also has important implications for empirical studies of auctions; in particular, the distribution of use values (or of bidders' expected use values at the auction) cannot be inferred from observed bids without detailed knowledge of the resale market structure. The information linkage between markets further complicates inference from bids and can also eliminate revenue equivalence in the primary market, introduce signaling strategies in the auction, and in some cases preclude existence of an equilibrium in symmetric separating (i.e., efficient) bidding strategies. However, the effects of this information linkage (including whether bidders want to signal high or low types) depend on the game played in the resale market, the type of auction held in the primary market, and the seller's bid announcement policy.²⁷

It should come as no surprise that a resale opportunity changes equilibrium bidding strategies; there are many cases in which embedding a simple game in a richer dynamic model changes equilibrium behavior. Nonetheless, the significance of the effects here are noteworthy, particularly since static auction models have often been applied to environments in which resale opportunities exist. While important insights from models of auctions without resale carry over, such models fail to capture important aspects of bidding at auctions with resale and can misguide the design of markets as well as the interpretation of bidding data.

Significant questions regarding auctions with resale opportunities remain open. The question of optimal auction design, in particular, has not been addressed here. Haile [14] studies the use of reserve prices in a simple version this model and shows that the endogeneity of bidder valuations can lead to equilibria in which some types to pool at the reserve price, making the derivation of an optimal reserve price considerably more complex and information intensive than in the standard model. Combined with the effects the primary market mechanism can have on the information available in the secondary market, this suggests significant challenges to the derivation of optimal selling mechanisms with resale.²⁸ Nonetheless, given the prevalence of resale opportunities, this is one important direction for future research.

²⁷Recent independent work by Goeree [10] considers the effects of signaling incentives in a related model.

²⁸Two recent papers consider this problem in different environments. Ausubel and Cramton [2] consider multi-unit auctions with resale, assuming an efficient secondary market. Zheng [36] focuses on an extension of Myerson's [29] auction design problem (where bidders have no private uncertainty) to the case in which optimal resale cannot be prevented.

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Appendix A. Proofs from Section 3

Proof of Lemma 2. $G_1(z|\mathbf{y}) = \prod_{i=1}^{n-1} G(z|y_i)$ which, by Lemma 1, strictly decreases in each y_i for all $z \in (0, 1)$. Similarly,

$$G_2(z|\mathbf{y}) = \sum_{i=1}^{n-1} \prod_{j \neq n,i} G(z|y_j) - (n-2) \prod_{i=1}^{n-1} G(z|y_i).$$

So for $k \in \{1, 2, \dots, n-1\}$

$$\begin{aligned} \frac{\partial G_2(z|\mathbf{y})}{\partial y_k} &= \left[\frac{\partial}{\partial y_k} G(z|y_k) \right] \left[\sum_{i \neq n,k} \left(\prod_{j \neq n,i,k} G(z|y_j) - \prod_{j \neq n,k} G(z|y_j) \right) \right] \\ &= \left[\frac{\partial}{\partial y_k} G(z|y_k) \right] \left[\sum_{i \neq n,k} \left((1 - G(z|y_i)) \prod_{j \neq n,i,k} G(z|y_j) \right) \right] < 0 \end{aligned}$$

with the inequality following from Lemma 1. \square

Proof of Theorem 1. We must verify that $b_{s,r}(\cdot)$ is strictly increasing and that the first-order condition characterizes an optimum. For the latter, it is sufficient to show that $\frac{\partial}{\partial \tilde{x}} \pi(x, \tilde{x})$ increases in x . Since

$$\frac{\partial}{\partial \tilde{x}} \pi(x, \tilde{x}) = [v_r(x, \tilde{x}) - b_{s,r}(\tilde{x})] f_1(\tilde{x}),$$

both results follow if $v_r(x, y)$ is strictly increasing in x and y for $r \in \{o, e\}$.

Letting

$$v(u, \mathbf{y}) = u G_1(u|\mathbf{y}) + \int_u^1 z_1 dG_1(z_1|\mathbf{y}),$$

we have

$$w_o(x, y) = E[v(U_i, \mathbf{Y}) | X_i = x, Y_1 = y].$$

Since $v(u, \mathbf{y})$ strictly increases in u , Lemma 1 implies that $w_o(x, y)$ strictly increases in x . Since $\ell_o(x, y) = 0 \forall x, y$, this implies that $v_o(x, y)$ also strictly increases in x . Integration by parts reveals that (by Lemma 2) $v(u, \mathbf{y})$ is strictly increasing in each y_j , ensuring that $w_o(x, y)$ (and, therefore, $v_o(x, y)$) is strictly increasing in y .

Now let

$$\hat{v}(u, \mathbf{y}) = u + \int_u^1 (z_2 - u) dG_2(z_2|\mathbf{y}),$$

so that

$$w_e(x, y) = E[\hat{v}(U_i, \mathbf{Y}) | X_i = x, Y_1 = y]$$

with

$$\frac{\partial}{\partial u} \hat{v}(u, \mathbf{y}) = G_2(u|\mathbf{y}).$$

Similarly, letting

$$\tilde{v}(u, \mathbf{y}) = \int_0^u (u - z_1) dG_1(z_1|\mathbf{y}),$$

we have

$$\ell_e(x, y) = E[\tilde{v}(U_i, \mathbf{Y}) | X_i = x, Y_1 = y]$$

with

$$\frac{\partial}{\partial u} \tilde{v}(u, \mathbf{y}) = G_1(u|\mathbf{y}).$$

Since $G_2(u|\mathbf{y}) > G_1(u|\mathbf{y}) \forall u \in (0, 1)$,

$$\frac{\partial}{\partial u} [\hat{v}(u, \mathbf{y}) - \tilde{v}(u, \mathbf{y})] > 0 \tag{A.1}$$

so that Lemma 1 ensures that $v_e(x, y)$ is strictly increasing in x . Finally, integration by parts shows that (by Lemma 2) $\hat{v}(u, \mathbf{y})$ is strictly increasing in \mathbf{y} and $\tilde{v}(u, \mathbf{y})$ is strictly decreasing in \mathbf{y} , ensuring that $v_e(x, y)$ is strictly increasing in y . \square

Proof of Theorem 2. Suppose $b(\cdot)$ is a differentiable equilibrium bid function. The payoff to a type x player bidding $b(\tilde{x})$ is

$$\pi(x, \tilde{x}) \equiv \int_0^{\tilde{x}} [w_r(x, y) - b(\tilde{x})] dF_1(y) + \int_{\tilde{x}}^1 \ell_r(x, y) dF_1(y).$$

The first-order condition gives a differential equation that, following standard arguments, can be shown to have unique solution

$$b_{f,r}(x) = \frac{\int_0^x V_r(t) dF_1(t)}{F_1(x)}.$$

Monotonicity of $V_r(\cdot)$ (shown in the proof of Theorem 1) ensures that this bid function is strictly increasing, while monotonicity of $v_r(x, y)$ in x (also shown in the

proof of Theorem 1) implies that $\frac{\partial}{\partial \tilde{x}} \pi(x, \tilde{x})$ increases in x , ensuring that the first-order condition characterizes an optimum. \square

Proof of Theorem 3. Suppose $b(\cdot; \mathbf{y}_{-1})$ is a symmetric separating equilibrium bid function for phase $n - 2$. Assuming equilibrium bidding by his opponents, a bidder with type x bidding $b(\tilde{x}; \mathbf{y}_{-1})$ expects payoff

$$\begin{aligned} \pi^{n-2}(x, \tilde{x}; \mathbf{y}_{-1}) &= \int_{y_2}^{\max\{\tilde{x}, y_2\}} [\hat{w}_r^{n-2}(x, y, \mathbf{y}_{-1}) - b(y; \mathbf{y}_{-1})] \frac{f(y)}{1 - F(y_2)} dy \\ &\quad + \int_{\max\{\tilde{x}, y_2\}}^1 \hat{v}_r^{n-2}(x, y, \mathbf{y}_{-1}) \frac{f(y)}{1 - F(y_2)} dy. \end{aligned} \tag{A.2}$$

The first-order condition yields

$$b(x; \mathbf{y}_{-1}) = \hat{v}_r^{n-2}(x, x, \mathbf{y}_{-1}).$$

Note that if $\hat{v}_r^{n-2}(x, y, \mathbf{y}_{-1})$ increases in x for all x, y, \mathbf{y}_{-1} , then $\frac{\partial}{\partial \tilde{x}} \pi^{n-2}(x, \tilde{x}; \mathbf{y}_{-1})$ increases in x , ensuring that the first-order condition gives an optimum. Using the function $v(u, \mathbf{y})$ defined in the proof of Theorem 1,

$$\hat{v}_o^{n-2}(x, y, \mathbf{y}_{-1}) = \hat{w}_o^{n-2}(x, y, \mathbf{y}_{-1}) = \int_0^1 v(u, \mathbf{y}) dG(u|x). \tag{A.3}$$

Since $v(u, \mathbf{y})$ strictly increases in u , Lemma 1 implies that $\hat{v}_o^{n-2}(x, y, \mathbf{y}_{-1})$ strictly increases in x . Similarly, using the functions $\hat{v}(u, \mathbf{y})$ and $\tilde{v}(u, \mathbf{y})$ defined in the proof of Theorem 1,

$$\hat{v}_e^{n-2}(x, y, \mathbf{y}_{-1}) = \int_0^1 [\hat{v}(u, \mathbf{y}) - \tilde{v}(u, \mathbf{y})] dG(u|x). \tag{A.4}$$

Hence inequality (A.1) and Lemma 1 imply strict monotonicity in x . \square

Proof of Theorem 4. From (A.3) and (A.4), we know each $\hat{b}_{e,r}^k(x; \mathbf{y}_{-1}^k)$ strictly increases in x and \mathbf{y}_{-1}^k for $r \in \{o, e\}$ if (i) $v(u, \mathbf{y})$ strictly increases in u and \mathbf{y} , (ii) $\hat{v}(u, \mathbf{y})$ strictly increases in \mathbf{y} , (iii) $\tilde{v}(u, \mathbf{y})$ strictly decreases in \mathbf{y} , and (iv) $\hat{v}(u, \mathbf{y}) - \tilde{v}(u, \mathbf{y})$ strictly increases in u . Each of these was shown in the proof of Theorem 1.

Let $\pi^k(x, \tilde{x}; \mathbf{y}_{-1}^k)$ denote the expected payoff of a bidder with type x who bids $b_{e,r}^k(\tilde{x}, \mathbf{y}_{-1}^k)$ in phase k . Taking $k \in \{0, \dots, n - 3\}$, suppose that in phase $k + 1$, for any \mathbf{y}_{-1}^{k+1}

$$\frac{\partial}{\partial \tilde{x}} \pi^{k+1}(x, \tilde{x}; \mathbf{y}_{-1}^{k+1}) \begin{cases} \geq 0 \quad \forall \tilde{x} < x \\ \leq 0 \quad \forall \tilde{x} > x. \end{cases} \tag{A.5}$$

Letting $\{y, \mathbf{y}_{-1}^k\}$ denote a feasible realization of \mathbf{Y}_{-1}^{k+1} given $\mathbf{Y}_{-1}^k = \mathbf{y}_{-1}^k$, for $k \leq n - 2$

$$\begin{aligned} \pi^k(x, \tilde{x}; \mathbf{y}_{-1}^k) &= \int_{y_{n-k}}^{\max\{\tilde{x}, y_{n-k}\}} \pi^{k+1}(x, x; \{y, \mathbf{y}_{-1}^k\}) d\hat{F}_k(y|\mathbf{y}_{-1}^k) \\ &\quad + \int_{\max\{\tilde{x}, y_{n-k}\}}^1 \hat{\zeta}_r^k(x, y, \mathbf{y}_{-1}^k) d\hat{F}_k(y|\mathbf{y}_{-1}^k) \end{aligned}$$

(where $Y_n \equiv 0$). Note that $\pi^{k+1}(x, x; \{y, \mathbf{y}_{-1}^k\})$ is the phase- $(k + 1)$ continuation value even after a deviation in phase k because (A.5) implies that if i reaches phase $k + 1$ by remaining in the auction past $b_{e,r}^k(x, \mathbf{y}_{-1}^k)$ in phase k , it will be optimal for him to exit immediately when phase $k + 1$ begins. The same argument implies that for all $t \geq x$

$$\pi^{k+1}(x, x; \{t, \mathbf{y}_{-1}^k\}) = \int_t^1 \hat{\zeta}_r^{k+1}(x, y, \{t, \mathbf{y}_{-1}^k\}) d\hat{F}_{k+1}(y|\{t, \mathbf{y}_{-1}^k\}). \tag{A.6}$$

Since by definition $\hat{\zeta}_r^k(x, \tilde{x}, \mathbf{y}_{-1}^k) = \int_{\tilde{x}}^1 \hat{\zeta}_r^{k+1}(x, y, \{\tilde{x}, \mathbf{y}_{-1}^k\}) d\hat{F}_{k+1}(y|\{\tilde{x}, \mathbf{y}_{-1}^k\})$, we can write

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}} \pi^k(x, \tilde{x}; \mathbf{y}_{-1}^k) &= \mathbf{1}\{\tilde{x} \geq y_{n-k}\} \hat{f}_k(\tilde{x}|\mathbf{y}_{-1}^k) \left[\pi^{k+1}(x, x; \{\tilde{x}, \mathbf{y}_{-1}^k\}) \right. \\ &\quad \left. - \int_{\tilde{x}}^1 \hat{\zeta}_r^{k+1}(x, y, \{\tilde{x}, \mathbf{y}_{-1}^k\}) d\hat{F}_{k+1}(y|\{\tilde{x}, \mathbf{y}_{-1}^k\}) \right]. \end{aligned}$$

For $\tilde{x} \geq x$ this equals zero by (A.6). When $\tilde{x} < x$ the first term in brackets is the expected payoff obtained by following the equilibrium strategy for phase $k + 1$ while the second term is that from exiting early at price $b_{e,r}^{k+1}(\tilde{x}; \{\tilde{x}, \mathbf{y}_{-1}^k\})$. Hence (A.5) implies that the bracketed expression cannot be negative when $\tilde{x} < x$. This gives

$$\frac{\partial}{\partial \tilde{x}} \pi^k(x, \tilde{x}; \mathbf{y}_{-1}^k) \begin{cases} \geq 0, & \tilde{x} < x \\ = 0, & \tilde{x} \geq x. \end{cases} \tag{A.7}$$

Optimality of the proposed sequence of bid functions then follows by induction, noting that the proof of Theorem 3 showed that for any \mathbf{y}_{-1}

$$\frac{\partial}{\partial \tilde{x}} \pi^{n-2}(x, \tilde{x}; \mathbf{y}_{-1}) \begin{cases} \geq 0 \quad \forall \tilde{x} < x \\ \leq 0 \quad \forall \tilde{x} > x. \end{cases} \quad \square$$

Proof of Theorem 5. Theorems 1 and 2 imply that bidding in the sealed-bid auctions is as if bidders had i.i.d. private values $\mathcal{V}_i = v_r(X_i, X_i)$ with distribution $\mathcal{F}(\cdot)$ such that $\mathcal{F}(v_r(x, x)) = F(x)$. Revenue equivalence of these auctions follows immediately from standard results (e.g., [31]). Now note that for all $x \geq y$

$$\begin{aligned} v_r(y, y) &= E[W_r(X_i, Y_1, \dots, Y_{n-1}) - L_r(X_i, Y_1, \dots, Y_{n-1}) | X_i = Y_1 = y] \\ &= E[W_r(Y_1, Y_1, \dots, Y_{n-1}) - L_r(Y_1, Y_1, \dots, Y_{n-1}) | X_i = x, Y_1 = y] \end{aligned}$$

and

$$\begin{aligned} \hat{v}_r^{n-2}(y, y, \mathbf{y}_{-1}) &= E[W_r(X_i, Y_1, \dots, Y_{n-1}) \\ &\quad - L_r(X_i, Y_1, \dots, Y_{n-1}) \mid X_i = Y_1 = y, \mathbf{Y}_{-1} = \mathbf{y}_{-1}] \\ &= E[W_r(Y_1, Y_1, \dots, Y_{n-1}) \\ &\quad - L_r(Y_1, Y_1, \dots, Y_{n-1}) \mid X_i = x, Y_1 = y, \mathbf{Y}_{-1} = \mathbf{y}_{-1}]. \end{aligned}$$

So, letting R_r^a denote the expected revenue from a first-stage auction of type a ,

$$\begin{aligned} R_r^s &= E[v_r(Y_1, Y_1) \mid X_i \geq Y_1] \\ &= E[\hat{v}_r^{n-2}(Y_1, Y_1, \mathbf{Y}_{-1}) \mid X_i \geq Y_1] = R_r^e. \quad \square \end{aligned}$$

Proof of Theorem 6. Theorem 5 implies that it is sufficient to compare revenues from a second-price auction with and without resale. Let R_{NR} denote the seller’s expected revenue with no resale market and R_r that with a resale market of type $r \in \{o, e\}$. For all x and y , $\ell_o(x, y) = 0$, while $w_o(x, y) > E[U_i \mid X_i = x]$. Let $X_{2,n}$ denote the second highest signal among all bidders and $f_{2,n}(\cdot)$ its density. Then we have

$$R_o = E[v_o(X_{2,n}, X_{2,n})] > E[E[U_i \mid X_i] \mid X_i = X_{2,n}] = R_{NR}.$$

Under EA, write $w_e(x, y; n)$, $\ell_e(x, y; n)$, and $v_e(x, y; n)$ to emphasize the dependence on the number of bidders. Likewise, let $G_{j(n)}(\cdot \mid \mathbf{Y})$ denote the distribution $G_j(\cdot \mid \mathbf{Y})$. I show (a) $v_e(x, x; n) < E[U_i \mid X_i = x] \forall x$ when $n = 2$; and (b) $\lim_{n \rightarrow \infty} (E[v_e(X_{2,n}, X_{2,n}; n)] - E[U_i \mid X_i = X_{2,n}]) > 0$. Since $R_e = E[v_e(X_{2,n}, X_{2,n}; n)]$ while $R_{NR} = E[E[U_i \mid X_i] \mid X_i = X_{2,n}]$, this will prove the result.

Observe that $w_e(x, y; 2) = E[U_i \mid X_i = x]$; since $\ell_e(x, x; 2) > 0 \forall x$, (a) follows. Define

$$\begin{aligned} \delta(x; n) &\equiv w_e(x, x; n) - E[U_i \mid X_i = x] \\ &= E\left[U_i + \int_{U_i}^1 (z_2 - U_i) dG_{2(n)}(z_2 \mid \mathbf{Y}) \mid X_i = x, Y_1 = x\right] \\ &\quad - E[U_i \mid X_i = x] \\ &= E\left[\int_{U_i}^1 (z_2 - U_i) dG_{2(n)}(z_2 \mid \mathbf{Y}) \mid X_i = x, Y_1 = x\right] \end{aligned} \tag{A.8}$$

which, for any $x \in [0, 1]$, is zero for $n = 2$ and strictly positive for $n \geq 3$. Furthermore, if we begin in an environment with $n \geq 3$ bidders and add a bidder with signal X_{n+1} , $G_{2(n)}(u \mid y_1, \dots, y_{n-1}) > G_{2(n+1)}(u \mid y_1, \dots, y_{n-1}, x_{n+1})$ for any $u \in (0, 1)$ regardless of the realization of X_{n+1} . This dominance relation implies that $\delta(x; n)$ is strictly increasing in n for all x . Therefore, $\lim_{n \rightarrow \infty} \delta(x; n) > 0 \forall x$, implying

$$\lim_{n \rightarrow \infty} \int_0^1 \delta(x; n) f_{2,n}(x) dx > 0. \tag{A.9}$$

Since

$$\ell_e(x, x; n) = E \left[\int_0^{U_i} (U_i - z_1) dG_{1(n)}(z_1 | \mathbf{Y}) | X_i = x, Y_1 = x \right]$$

and

$$\lim_{n \rightarrow \infty} G_{1(n)}(u | \mathbf{y}) = \lim_{n \rightarrow \infty} \prod_{j=1}^{n-1} G(u | y_j) = 0 \quad \forall u \in [0, 1)$$

$\lim_{n \rightarrow \infty} \ell_e(x, x; n) = 0$ for all x . This implies

$$\lim_{n \rightarrow \infty} \int_0^1 \ell_e(x, x; n) f_{2:n}(x) dx = 0$$

which, with (A.9), gives (b). \square

Appendix B. Resale outcomes with asymmetric information

B.1. Second-stage optimal auction

In the optimal resale auction [29], each first-stage loser j is asked to report his use value, from which the seller calculates the “virtual use value”

$$c_j = c(\tilde{u}_j, \tilde{x}_j) = \tilde{u}_j - \frac{1 - G(\tilde{u}_j | \tilde{x}_j)}{g(\tilde{u}_j | \tilde{x}_j)},$$

where \tilde{u}_j is j 's report and \tilde{x}_j is the signal inferred from j 's first-stage bid. The first-stage winner, player i say, sells to the player $h \neq i$ with the highest virtual use value, c_h , if and only if $c_h > u_i$. If the sale occurs, h pays a price equal to the lowest use value he could have had and still have been chosen to buy in the second stage, i.e.,

$$c^{-1}(\max\{u_i, \hat{c}_1\}; \tilde{x}_h)$$

where \hat{c}_1 is the highest virtual use value among all other second-stage buyers and (4) ensures that $c(\cdot; \tilde{x}_h)$ is invertible. Because j 's announcement does not affect the price he pays and for any \hat{c}_1 , u_i , and \tilde{x}_j

$$\{u_j > c^{-1}(\max\{u_i, \hat{c}_1\}; \tilde{x}_j)\} \Leftrightarrow \{c(u_j; \tilde{x}_j) > \max\{u_i, \hat{c}_1\}\}$$

announcing $\tilde{u}_j = u_j$ is a dominant strategy, even when $\tilde{x}_j \neq x_j$.

Myerson's equation (4.12) implies that the expected resale profit to the second-stage seller is equal to the probability of a sale occurring multiplied by the expected virtual use value of the buyer conditional on a sale. This resale profit is added to the first-stage winner's expected use value to obtain his expected gross payoff. Let

$$H(\hat{c}_1 | \mathbf{y}_{-1}) = \prod_{j=2}^{n-1} G(c^{-1}(\hat{c}_1; y_j) | y_j)$$

denote the distribution of the highest virtual use value among i 's opponents, excluding the opponent with first-stage signal Y_1 . Then for $y \leq \tilde{x}$

$$\begin{aligned}
 w_o(x, y, \tilde{x}) = E & \left[\int_0^1 \left\{ uG(c^{-1}(u; y)|y)H(u|\mathbf{Y}_{-1}) \right. \right. \\
 & + \int_{c^{-1}(u; y)}^1 c(t, y)H(c(t, y)|\mathbf{y}_{-1}) dG(t|y) \\
 & \left. \left. + \int_u^1 \hat{c}_1 G(c^{-1}(\hat{c}_1; y)|y) dH(\hat{c}_1|\mathbf{Y}_{-1}) \right\} dG(u|x) | Y_1 = y \right]. \quad (\text{B.1})
 \end{aligned}$$

The first term reflects the case in which no second-stage buyer's virtual use value exceeds the first-stage winner's use value (so that no sale occurs), the second two terms the case in which a sale does occur—to the buyer with signal $Y_1 = y$ in the first term and to some other buyer in the last term. The winner's own first-stage bid plays no role in the outcome of the resale auction; therefore, \tilde{x} does not appear on the right-hand side of (B.1).

Now consider the expected payoff of a losing bidder. Let w denote the index of the winning bidder. For $y \geq \tilde{x}$ we can define $\ell_o(x, y, \tilde{x})$ as

$$\begin{aligned}
 E & \left[\int_0^1 \int_{c(0,1)}^{c(u, \tilde{x})} \int_0^{c(u, \tilde{x})} (u - c^{-1}(\max\{u_w, \hat{c}_1\}; \tilde{x})) dG(u_w|y) \right. \\
 & \left. \times dH(\hat{c}_1|\mathbf{Y}_{-1}) dG(u|x) | Y_1 = y \right]. \quad (\text{B.2})
 \end{aligned}$$

Here a player's first-stage bid affects his second-stage payoff because it determines the distribution function used to construct his virtual use value. Since $c(u, x)$ is strictly increasing and differentiable in u (by (4)) and strictly decreasing and differentiable in x (by Lemma 1 and the differentiability of $g(u|x)$ with respect to x), $c^{-1}(c; x)$ is strictly increasing and differentiable in x . Hence, $D_3[\ell_o(x, y, \tilde{x})]$ exists and is equal to

$$\begin{aligned}
 - E & \left[\int_0^1 \int_{c(0,1)}^{c(u, \tilde{x})} \int_0^{c(u, \tilde{x})} \frac{\partial}{\partial \tilde{x}} c^{-1}(\max\{u_w, \hat{c}_1\}; \tilde{x}) dG(u_w|y) \right. \\
 & \left. \times dH(\hat{c}_1|\mathbf{Y}_{-1}) dG(u|x) | Y_1 = y \right] < 0.
 \end{aligned}$$

Expressions for $\hat{w}_0^k(x, y, \mathbf{y}_{-1}^k, \tilde{x})$ and $\hat{\ell}_0^k(x, y, \mathbf{y}_{-1}^k, \tilde{x})$ are identical to (B.1) and (B.2) except that expectations are taken over $\mathbf{Y}_{-1} \setminus \mathbf{Y}_{-1}^k$ conditional on $Y_{n-k-1} = y$ and $\mathbf{Y}_{-1}^k = \mathbf{y}_{-1}^k$. Hence the arguments above imply that $D_4[\hat{w}_0^k(x, y, \mathbf{y}_{-1}^k, \tilde{x})] = 0$ and $D_4[\hat{\ell}_0^k(x, y, \mathbf{y}_{-1}^k, \tilde{x})] < 0$.

B.2. Second-stage English auction

Under EA, as long as at least two bidders remain in the second-stage auction, each has a weakly dominant strategy of remaining as long as the price is below his use value. Once only a single bidder remains, he must choose a final offer to the resale seller that can be no lower than the price p at which his last opponent dropped out. Let $G^w(\cdot|\alpha^a)$ denote the first-stage losers' common posterior on the winner's use value, based on the bids α^a announced after an auction of type a . If the last bidder in the second stage has use value u , his optimal ultimatum offer solves

$$\max_{b \geq p} (u - b)G^w(b|\alpha^a).$$

The unique solution is $b^* = \max\{p, B_*(u; \alpha^a)\}$, with $B_*(u; \alpha^a)$ defined implicitly by the equation

$$B_* = u - \frac{G^w(B_*|\alpha^a)}{g^w(B_*|\alpha^a)}. \tag{B.3}$$

The index a on α^a is important. When a sealed bid auction is held in the first stage, the winner's bid $b(\tilde{x})$ will be included in α^a , implying that $G^w(u|\alpha^a) = G(u|\tilde{x})$. In this case I denote $B_*(u; \alpha^s)$ by $B_*^s(u; \tilde{x})$. When an English auction is held in the first stage, the winner's bid is not revealed, so $G^w(\cdot|\alpha^e)$ will be constructed based only on the inference that the first-stage winner's signal must have been higher than that of the marginal bidder in the first stage. In particular, if the next-to-last bidder's type y is inferred from his exit price, $G^w(u|\alpha^e) = \int_y^1 G(u|x) \frac{f(x)}{1-F(y)} dx$. In this case I denote $B_*(u; \alpha^e)$ by $B_*^e(u; y)$.

Remark B.1. $B_*^s(u; y) < B_*^e(u; y) \forall y \in (0, 1), u \in (0, 1)$.

Proof. Strict affiliation implies

$$g(t|y)g(b|x) > g(b|y)g(t|x) \quad \forall b > t, x > y.$$

Integrating both sides over $[0, b]$ with respect to t and then over $[y, 1]$ with respect to $F(x)$ yields

$$\frac{G(b|y)}{g(b|y)} > \frac{\int_y^1 G(b|x) dF(x)}{\int_y^1 g(b|x) dF(x)}.$$

The result then follows from (B.3), (5) and (6). \square

For $a \in \{s, e\}$ the lowest second-stage type whose unconstrained offer is at least u is

$$u_*^a(u; t) = \inf \{ \hat{u} \in [0, 1] : B_*^a(\hat{u}; t) \geq u \}.$$

Then for a first-stage sealed bid auction we can derive (for $y \leq \tilde{x}$)

$$\begin{aligned}
 &w_e(x, y, \tilde{x}) \\
 &= E \left[\int_0^1 \left\{ u + \int_{u^s(u; \tilde{x})}^{u^s_*(u; \tilde{x})} \int_u^{z_1} (z_2 - u) dG_2(z_2 | \mathbf{Y}, z_1) dG_1(z_1 | \mathbf{Y}) \right. \right. \\
 &\quad + \int_{u^s(u; \tilde{x})}^1 \int_{B_*^s(z_1; \tilde{x})}^{z_1} (z_2 - u) dG_2(z_2 | \mathbf{Y}, z_1) dG_1(z_1 | \mathbf{Y}) \\
 &\quad \left. \left. + \int_{u^s_*(u; \tilde{x})}^1 \int_0^{B_*^s(z_1; \tilde{x})} (B_*^s(z_1; \tilde{x}) - u) dG_2(z_2 | \mathbf{Y}, z_1) dG_1(z_1 | \mathbf{Y}) \right\} \right. \\
 &\quad \left. dG(u|x) | Y_1 = y \right]. \tag{B.4}
 \end{aligned}$$

The first integral inside the braces gives the contribution to the first-stage winner i 's expected resale profit from the possibility that $Z_1 < u^s_*(U_i; \tilde{x})$ but $Z_2 > U_i$; the second integral gives the contribution from the event $\{Z_1 > u^s_*(U_i; \tilde{x}) \text{ and } Z_2 > B_*^s(Z_1; \tilde{x})\}$. The final integral gives the contribution from the case that $Z_1 > u^s_*(U_i; \tilde{x})$ and $Z_2 < B_*^s(Z_1; \tilde{x})$, so that $B_*^s(Z_1; \tilde{x})$, not Z_2 , is the resale price. The expression for $\hat{w}_e^k(x, y, \mathbf{y}_{-1}^k, \tilde{x})$ is identical to (B.4) except that $B_*^e(\cdot; y)$ and $u_*^e(\cdot; y)$ replace $B_*^s(\cdot; \tilde{x})$ and $u_*^s(\cdot; \tilde{x})$, respectively, and the expectation is taken conditional on $Y_{n-k-1} = y$ and $\mathbf{Y}_{-1}^k = \mathbf{y}_{-1}^k$. Note that with these changes, however, \tilde{x} does not appear.

Differentiating (B.4) with respect to \tilde{x} gives

$$\begin{aligned}
 &D_3[w_e(x, y, \tilde{x})] \\
 &= E \left[\int_0^1 \int_{u^s_*(u; \tilde{x})}^{u^s_*(u; \tilde{x})} \frac{\partial B_*^s(z_1; \tilde{x})}{\partial \tilde{x}} G_2(B_*^s(z_1; \tilde{x}) | \mathbf{Y}, z_1) \right. \\
 &\quad \left. \times dG_1(z_1 | \mathbf{Y}) dG(u|x) | Y_1 = y \right]. \tag{B.5}
 \end{aligned}$$

Lemma 1 and (5) imply that $B_*^s(z_1; \tilde{x})$ strictly increases in \tilde{x} . Given the assumed differentiability of $G(u|x)$ and $g(u|x)$, this implies that the derivative in (B.5) exists and is positive.

A bidder who loses the first-stage auction buys in the secondary market if he has the highest use value among the losers and either (a) his own use value is high enough that he is willing to make an ultimatum offer the first-stage winner will accept or (b) the second highest use value is above that of the first-stage winner. Let $G_{-1}(\cdot | \mathbf{y}_{-1})$ give the conditional distribution of the highest use value among a given bidder's opponents, excluding the opponent with the highest first-stage signal Y_1 . Then the expected payoff to bidder i when he loses a sealed bid auction can be

written (for $y \geq \tilde{x}$)

$$\begin{aligned} \ell_e(x, y, \tilde{x}) = E \left[\int_0^1 \left\{ [u - B_*^s(u; y)] G(B_*^s(u; y)|y) G_{-1}(B_*^s(u; y)|\mathbf{Y}_{-1}) \right. \right. \\ \left. \left. + \int_{B_*^s(u; y)}^u (u - t) G(t|y) dG_{-1}(t|\mathbf{Y}_{-1}) \right\} dG(u|x) | Y_1 = y \right]. \end{aligned} \quad (\text{B.6})$$

Since beliefs about buyer’s types do not affect the resale auction, \tilde{x} does not appear.

The expected payoff to a loser of a first-stage English auction is complicated slightly by the fact that when i has the second highest bid in the first stage, all losers will use his bid to infer a lower bound on the first-stage winner’s signal. Otherwise, $\hat{\gamma}_e^k(x, y, \mathbf{y}_{-1}, \tilde{x})$ is as in (B.6):

$$\begin{aligned} E \left[\int_0^1 \left\{ \mathbf{1}\{\tilde{x} \geq Y_2\} \left[G_1(B_*^e(u; \tilde{x})|\mathbf{Y}) [u - B_*^e(u; \tilde{x})] \right. \right. \right. \\ \left. \left. + \int_{B_*^e(u; \tilde{x})}^u (u - t) G(t|Y_1) dG_{-1}(t|\mathbf{Y}_{-1}) \right] \right. \\ \left. + \mathbf{1}\{Y_2 > \tilde{x}\} \left[G_1(B_*^e(u; Y_2)|\mathbf{y}) [u - B_*^e(u; Y_2)] \right. \right. \\ \left. \left. + \int_{B_*^e(u; Y_2)}^u (u - t) G(t|Y_1) dG_{-1}(t|\mathbf{Y}_{-1}) \right] \right\} dG(u|x) \right] \end{aligned} \quad (\text{B.7})$$

with the expectation taken over $\mathbf{Y}_{-1} \setminus \mathbf{Y}_{-1}^k$ conditional on $Y_{n-1-k} = y$ and $\mathbf{Y}_{-1}^k = \mathbf{y}_{-1}^k$. The first line gives the contribution to the expected payoff when $b(\tilde{x})$ is the marginal bid in the first stage, while the second line gives the contribution from the case $b(\tilde{x}) < b(y_2)$.

Remark B.2. $\hat{v}_e^{n-2}(x, y, \mathbf{y}_{-1}, y)$ and $\hat{v}_e^{n-2}(x, x, \mathbf{y}_{-1}, x)$ strictly increase in x for all $x \in (0, 1)$.

Proof. Strict monotonicity of $\hat{v}_e^{n-2}(x, y, \mathbf{y}_{-1}, y)$ in x is shown by Haile [15, Lemma 1]. Monotonicity of $\hat{v}_e^{n-2}(x, y, \mathbf{y}_{-1}, x)$ in y follows in a straightforward manner from affiliation, since the payoff to the winner (loser) is weakly increasing (decreasing) in the use values of others. The proof is then completed by showing that $\hat{v}_e^{n-2}(x, x, \mathbf{y}_{-1}, \tilde{x})$ is increasing in \tilde{x} at $\tilde{x} = x$. Differentiating (B.7) with respect to \tilde{x} shows that $\frac{\partial}{\partial \tilde{x}} \hat{\gamma}_e^{n-2}(x, x, \mathbf{y}_{-1}, \tilde{x})$ equals

$$\begin{aligned} \int_0^1 \frac{\partial B_*^e(u; \tilde{x})}{\partial \tilde{x}} G_{-1}(B_*^e(u; \tilde{x})|\mathbf{y}_{-1}) \left[(u - B_*^e(u; \tilde{x})) g(B_*^e(u; \tilde{x})|x) \right. \\ \left. - G(B_*^e(u; \tilde{x})|x) \right] dG(u|x). \end{aligned}$$

Remark B.1 and (B.3) imply that the term in brackets is nonpositive at $\tilde{x} = x < 1$. Since $\frac{\partial B_e^*(u; \tilde{x})}{\partial \tilde{x}} \geq 0$ and $D_4[\hat{\theta}_e^{n-2}(x, x, \mathbf{y}_{-1}, \tilde{x})] = D_4[\hat{\theta}_e^{n-2}(x, x, \mathbf{y}_{-1}, \tilde{x})]$, this proves the result. \square

Appendix C. Proofs from Section 4

Proof of Theorem 8. Assuming a strictly increasing, differentiable equilibrium bid function $b(\cdot)$, a bidder with type x chooses \tilde{x} to maximize

$$\int_0^{\tilde{x}} [w_r(x, y, \tilde{x}) - b(\tilde{x})] dF_1(y) + \int_{\tilde{x}}^1 \ell_r(x, y, \tilde{x}) dF_2(y).$$

The first-order condition gives a differential equation that, following standard arguments, can be shown to have unique solution $b(x) = \frac{\int_0^x \theta_r(t) dF_1(t)}{F_1(x)}$. \square

Proof of Lemma 3. The result is immediate under OA, so assume EA defines the resale market. Since a first-stage winner benefits when his second-stage opponents believe he has a higher type, any equilibrium bid function $\hat{b}(\cdot; \mathbf{y})$ for phase $n - 1$ must be weakly increasing in the final bidder’s signal. Furthermore, wherever $\hat{b}(\cdot; \mathbf{y})$ is strictly increasing, it must be continuous, since otherwise some bidder would be able to lower his phase $n - 1$ bid by a discrete amount while changing the beliefs about his type (and therefore his payoff in the resale market) continuously. Suppose that $\hat{b}(\cdot; \mathbf{y})$ is both continuous and strictly increasing on $[y_1, 1]$. A bidder with signal $x > y_1$ chooses \tilde{x} to maximize

$$\tilde{\pi}(x, \tilde{x}) = \tilde{w}_e(x, y_1, \mathbf{y}_{-1}, \tilde{x}) - \hat{b}(\tilde{x}; \{y_1; \mathbf{y}_{-1}\}),$$

where \tilde{w}_e incorporates the modification of play in the second stage resulting from the fact that the winner’s type can be inferred from his bid—i.e., $\tilde{w}_e(x, y_1, \mathbf{y}_{-1}, \tilde{x})$ is given by the right-hand side of (B.4) with \mathbf{y}_{-1} taken as known. Separation by the type- x winner requires

$$D_{21}[\tilde{\pi}(x, x)] \geq 0$$

since otherwise types just below x would benefit from mimicking x ’s phase $n - 1$ bid. Differentiating (B.4) with \mathbf{y}_{-1} taken as known gives

$$\begin{aligned} D_2[\tilde{\pi}(x, \tilde{x})] &= \int_0^1 \int_{u_e^*(u; \tilde{x})}^1 \frac{\partial B_e^*(z_1; \tilde{x})}{\partial \tilde{x}} G_2(B_e^*(z_1; \tilde{x}) | \mathbf{y}, z_1) \\ &\quad \times dG_1(z_1 | \mathbf{y}) dG(u | x) - \hat{b}'(\tilde{x}) \\ &\equiv \int_0^1 \Omega(u, \tilde{x}) dG(u | x) - \hat{b}'(\tilde{x}). \end{aligned}$$

At $\tilde{x} = x$

$$\frac{\partial \Omega(u, x)}{\partial u} = -\frac{\partial u_*^s(u; x)}{\partial u} g_1(u_*^s(u; x) | \mathbf{y}) G_2(u | \mathbf{y}, u_*^s(u; x)) \frac{\partial}{\partial \tilde{x}} B_*^s(u_*^s(u; x); \tilde{x}) \Big|_{\tilde{x}=x}.$$

For all $u \in (0, 1)$, $\frac{\partial u_*^s(u; x)}{\partial u} > 0$ by (5). Since $B_*^s(z; \tilde{x})$ strictly increases in \tilde{x} , this and Lemma 1 imply that $D_{21}[\hat{\pi}(x, x)] < 0$. Hence, full separation is impossible.

A similar argument rules out partial separation in phase $n - 1$. If $\hat{b}(x; \mathbf{y}) = \hat{b} \forall x \in [x_L, x_U]$ but $\hat{b}(x; \mathbf{y}) > \hat{b} \forall x > x_U$, then for all $\varepsilon > 0$ we must have $\hat{\pi}(x_U + \varepsilon, x_U + \varepsilon) \geq \hat{\pi}(x_U + \varepsilon, x_U)$. But then since $D_{21}[\hat{\pi}(x, x)] < 0 \forall x$, $\hat{\pi}(x_U, x_U + \varepsilon) > \hat{\pi}(x_U, x_U)$ for sufficiently small ε . Analogous arguments rule out any form of partial separation. Hence, all types must pool in phase $n - 1$. \square

Proof of Lemma 4. It is sufficient to show that for all $x \in (0, 1)$

$$w_e(x, x, x) < E[\hat{w}_e^{n-2}(X_i, X_i, \mathbf{Y}_{-1}, X_i) | X_i = x] \tag{C.1}$$

and

$$l_e(x, x, x) \geq E[\hat{l}_e^{n-2}(X_i, X_i, \mathbf{Y}_{-1}, X_i) | X_i = x]. \tag{C.2}$$

Inequality (C.2) is immediate from the fact that following a sealed bid auction the top resale buyer will know $Y_1 = x$ when choosing his ultimatum offer, while following an English auction he will know only $Y_1 \geq x$. Remark B.1 implies that for all $x \in (0, 1)$

$$\begin{aligned} w_e(x, x, x) &= E[\max\{U_i, B_*^s(Z_1; X_i), Z_2\} | X_i = x, Y_1 = x] \\ &< E[\max\{U_i, B_e^c(Z_1; X_i), Z_2\} | X_i = x, Y_1 = x] \\ &= E[w_e^{n-2}(X_i, X_i, \mathbf{Y}_{-1}, X_i) | X_i = x] \end{aligned}$$

implying (C.1). \square

Proof of Theorem 9. Suppose bidder i 's opponents follow strategies \tilde{b}^k satisfying (i)–(iii) in each phase k . The pooling strategy $\tilde{b}_{e,e}^{n-1}$ is easily supported by beliefs off the equilibrium path (for example, specifying the same beliefs about the winner's type regardless of his bid). Given that $\tilde{b}_{e,e}^{n-1}(x; \{y, \mathbf{y}_{-1}\}) = \tilde{b}_{e,e}^{n-2}(y; \mathbf{y}_{-1})$, in phase $n - 2$ bidder i chooses \tilde{x} to maximize

$$\begin{aligned} \pi^{n-2}(x, \tilde{x}; \mathbf{y}_{-1}) &= \int_{y_2}^{\max\{y_2, \tilde{x}\}} [\hat{w}_e^{n-2}(x, y, \mathbf{y}_{-1}, \tilde{x}) - \tilde{b}_{e,e}^{n-2}(y; \mathbf{y}_{-1})] dF_1(y) \\ &\quad + \int_{\max\{y_2, \tilde{x}\}}^1 \hat{l}_e^{n-2}(x, y, \mathbf{y}_{-1}, \tilde{x}) dF_1(y). \end{aligned}$$

Recalling (B.3) and Remark B.2 (see [15]), differentiation shows that $D_{21}[\pi^{n-2}(x, \tilde{x}; \mathbf{y}_{-1})] \geq 0$. Hence

$$D_2[\pi^{n-2}(x, \tilde{x}; \mathbf{y}_{-1})] \begin{cases} \leq 0, & \tilde{x} > x \\ \geq 0, & \tilde{x} \leq x. \end{cases}$$

The induction argument used to prove Theorem 4 can then be applied to complete the proof. \square

Proof of Theorem 10. Suppose there were a symmetric separating equilibrium. Consider a bidder with type x who bids as type \tilde{x} in phase $n - 3$ after following equilibrium bidding strategies to this point. If $y_2 < \min\{x, \tilde{x}\}$, he will not exit in this phase, and his continuation payoff in phase $n - 2$ will be on the equilibrium path and equal to $\pi^{n-2}(x, x; \{y_2, \mathbf{y}_{-1}^{n-3}\})$. If $\tilde{x} > x$ and $y_2 \in [x, \tilde{x}]$, he also continues to phase $n - 2$ but obtains some continuation payoff $\gamma^{n-2}(x; \{y_2, \mathbf{y}_{-1}^{n-3}\})$, which is off the equilibrium path except when $y_2 = x$. Hence, this player chooses $\tilde{x} \geq y_3$ to maximize

$$\begin{aligned} \pi^{n-3}(x, \tilde{x}; \mathbf{y}_{-1}^{n-3}) &= \int_{y_3}^{\min\{x, \tilde{x}\}} \pi^{n-2}(x, x; \{y, \mathbf{y}_{-1}^{n-3}\}) d\hat{F}_{n-3}(y|\mathbf{y}_{-1}^{n-3}) \\ &\quad + \int_x^{\max\{x, \tilde{x}\}} \gamma^{n-2}(x; \{y, \mathbf{y}_{-1}^{n-3}\}) d\hat{F}_{n-3}(y|\mathbf{y}_{-1}^{n-3}) \\ &\quad + \int_{\tilde{x}}^1 \hat{\gamma}_0^{n-3}(x, y, \mathbf{y}_{-1}^{n-3}, \tilde{x}) d\hat{F}_{n-3}(y|\mathbf{y}_{-1}^{n-3}). \end{aligned}$$

Differentiating with respect to \tilde{x} gives

$$\begin{aligned} &\left[\frac{1}{\hat{f}_{n-3}(\tilde{x}|\mathbf{y}_{-1}^{n-3})} \right] \frac{\partial}{\partial \tilde{x}} \pi^{n-3}(x, \tilde{x}; \mathbf{y}_{-1}^{n-3}) \\ &= \mathbf{1}\{\tilde{x} < x\} \pi^{n-2}(x, x; \{\tilde{x}, \mathbf{y}_{-1}^{n-3}\}) \\ &\quad + \mathbf{1}\{\tilde{x} \geq x\} \gamma^{n-2}(x; \{\tilde{x}, \mathbf{y}_{-1}\}) - \hat{\gamma}_0^{n-3}(x, \tilde{x}, \mathbf{y}_{-1}^{n-3}, \tilde{x}) \\ &\quad + \int_{\tilde{x}}^1 D_4[\hat{\gamma}_0^{n-3}(x, y, \mathbf{y}_{-1}^{n-3}, \tilde{x})] \frac{\hat{f}_{n-3}(y|\mathbf{y}_{-1}^{n-3})}{\hat{f}_{n-3}(\tilde{x}|\mathbf{y}_{-1}^{n-3})} dy. \end{aligned} \tag{C.3}$$

Since $\gamma^{n-2}(x; \{x, \mathbf{y}_{-1}^{n-3}\})$ is on the equilibrium path, it must equal $\pi^{n-2}(x, x; \{x, \mathbf{y}_{-1}^{n-3}\})$. For $y_2 \leq x$

$$\begin{aligned} &\pi^{n-2}(x, x; \{y_2, \mathbf{y}_{-1}^{n-3}\}) \\ &= \int_{y_2}^x [\hat{w}_0^{n-2}(x, y, \{y_2, \mathbf{y}_{-1}^{n-3}\}, x) - b^{n-2}(y; \{y_2, \mathbf{y}_{-1}^{n-3}\})] \frac{f(y)}{1 - F(y_2)} dy \\ &\quad + \int_x^1 \hat{\gamma}_0^{n-2}(x, y, \{y_2, \mathbf{y}_{-1}^{n-3}\}, x) \frac{f(y)}{1 - F(y_2)} dy, \end{aligned}$$

where $b^{n-2}(\cdot; \mathbf{y}_{-1}^{n-2})$ is the equilibrium bid function for phase $n - 2$. This implies that

$$\begin{aligned} \pi^{n-2}(x, x; \{x, \mathbf{y}_{-1}^{n-3}\}) &= \int_x^1 \hat{\ell}_0^{n-2}(x, y, \{x, \mathbf{y}_{-1}^{n-3}\}, x) \frac{f(y)}{1 - F(x)} dy \\ &\equiv \hat{\ell}_0^{n-3}(x, x, \mathbf{y}_{-1}^{n-3}, x). \end{aligned}$$

Hence, the first term on the right-hand side of (C.3) drops out at $\tilde{x} = x$ while the second and third terms cancel. Since $D_4[\hat{\ell}_0^{n-3}(x, y, \mathbf{y}_{-1}^{n-3}, x)] < 0$, this implies that (C.3) cannot equal zero at $\tilde{x} = x < 1$. \square

Appendix D. “Final bid” equilibria in the English auction

Consider an English auction when the resale market structure is EA. Lemma 3 shows that all types pool in phase $n - 1$, and Theorem 9 characterizes the set of equilibria in which this pooling occurs at the previous exit price. This appendix briefly discusses other equilibria. Each can be viewed as a payoff-irrelevant perturbation of an equilibrium characterized by Theorem 9: the next-to-last bidder, j say, exits at a lower price, but the final bidder is forced by beliefs off the equilibrium path to make the bid that j would have made in the original equilibrium.

Let $\mathcal{P}(Y_1, \mathbf{Y}_{-1})$ denote the equilibrium pooling price for phase $n - 1$ given Y_1 and \mathbf{Y}_{-1} . For simplicity assume that for all \mathbf{y}_{-1} $\mathcal{P}(\cdot, \mathbf{y}_{-1})$ is a continuous function. Suppose that for some y and \mathbf{y}_{-1} , $\mathcal{P}(y, \mathbf{y}_{-1}) > \hat{v}_e^{n-2}(y, y, \mathbf{y}_{-1}, y)$. Then let a bidder with type $y + \varepsilon$ deviate to type y 's equilibrium bid in phase $n - 2$. This affects his payoff only if his phase- $(n - 2)$ opponent's type x lies in $(y, y + \varepsilon)$, which occurs with positive probability. When this does occur, he receives $\hat{\ell}_e^{n-2}(y + \varepsilon, x, \mathbf{y}_{-1}, y)$ instead of $\hat{w}_e^{n-2}(y + \varepsilon, x, \mathbf{y}_{-1}, y + \varepsilon) - \mathcal{P}(x, \mathbf{y}_{-1})$. Continuity of $\hat{v}_e^{n-2}(x, y, \mathbf{y}_{-1}, \tilde{x})$ and $\mathcal{P}(x, \mathbf{y}_{-1})$ in x, y and \tilde{x} implies that the deviation is profitable for sufficiently small ε . So suppose instead that $\mathcal{P}(y, \mathbf{y}_{-1}) < \hat{v}_e^{n-2}(y, y, \mathbf{y}_{-1}, y)$ for some y . Then for small ε a bidder with type $y - \varepsilon$ would prefer to make type y 's equilibrium bid in phase $n - 2$ since, when the deviation has any effect, he could obtain $\hat{w}_e^{n-2}(y - \varepsilon, x, \mathbf{y}_{-1}, y) - \mathcal{P}(x, \mathbf{y}_{-1})$ instead of $\hat{\ell}_e^{n-2}(y - \varepsilon, x, \mathbf{y}_{-1}, y - \varepsilon)$ for some $x \in (y - \varepsilon, y)$. Hence we must have $\mathcal{P}(y, \mathbf{y}_{-1}) = \hat{v}_e^{n-2}(y, y, \mathbf{y}_{-1}, y)$ for all y and \mathbf{y}_{-1} .²⁹

Now assume temporarily that even off the equilibrium path the winning bidder is always willing pool at $\mathcal{P}(Y_1, \mathbf{Y}_{-1})$ in phase $n - 1$. Then a type- x bidder in phase $n - 2$

²⁹Without continuity of $\mathcal{P}(\cdot, \mathbf{y}_{-1})$, a similar argument shows that $\mathcal{P}(y, \mathbf{y}_{-1}) = \hat{v}_e^{n-2}(y, y, \mathbf{y}_{-1}, y)$ almost everywhere, which is sufficient for what follows.

chooses \tilde{x} to maximize

$$\int_{y_2}^{\max\{y_2, \tilde{x}\}} [\hat{w}_e^{n-2}(x, y, \mathbf{y}_{-1}, x) - \hat{v}_e^{n-2}(y, y, \mathbf{y}_{-1}, y)] d\hat{F}_{n-2}(y|\mathbf{y}_{-1}) + \int_{\max\{y_2, \tilde{x}\}}^1 \hat{\ell}_e^{n-2}(x, y, \mathbf{y}_{-1}, x) d\hat{F}_{n-2}(y|\mathbf{y}_{-1}).$$

Since $\hat{v}_e^{n-2}(x, x, \mathbf{y}_{-1}, x)$ is strictly increasing in x (Remark B.2), $\tilde{x} = x$ is optimal; i.e., this bidder is willing to follow any strictly increasing bid function $b^{n-2}(\cdot; \mathbf{y}_{-1})$ used by all his opponents as long as $b^{n-2}(x; \mathbf{y}_{-1}) \leq \hat{v}_e^{n-2}(x, X, \mathbf{y}_{-1}, x)$.

The requirement that the winner always (along the equilibrium path) be willing to make the equilibrium pooling bid in phase $n - 1$ does place an additional restriction on the bid function. To obtain the largest set of equilibrium outcomes, assume any deviation in phase $n - 1$ of the first stage causes resale buyers to place probability one on the resale seller's use value being zero—the most punitive beliefs possible. Then the optimal deviation in phase $n - 1$ is to exit immediately, and the payoff to the deviant winner is

$$E[\max\{U_i, Z_2\} | X_i = x, \mathbf{Y} = \{y_1, \mathbf{y}_{-1}\}] - b^{n-2}(y_1, \mathbf{y}_{-1}).$$

Deviation in phase $n - 1$ by a type- x bidder is then always unprofitable if for all y and \mathbf{y}_{-1}

$$\hat{w}_e^{n-2}(x, y, \mathbf{y}_{-1}, y) - \hat{v}_e^{n-2}(y, y, \mathbf{y}_{-1}, y) \geq E[\max\{U_i, Z_2\} | X_i = x, \mathbf{Y} = \{y, \mathbf{y}_{-1}\}] - b^{n-2}(y, \mathbf{y}_{-1}). \tag{D.4}$$

We can write $\hat{w}_e^{n-2}(x, y, \mathbf{y}_{-1}, y) - E[\max\{U_i, Z_2\} | X_i = x, \mathbf{Y} = \{y, \mathbf{y}_{-1}\}]$ as

$$E[\bar{v}(U_i, \mathbf{Y}) - \hat{v}(U_i, \mathbf{Y}) | X_i = x, \mathbf{Y} = \{y, \mathbf{y}_{-1}\}]$$

with $\bar{v}(U_i, \mathbf{Y})$ given by the expression in braces in (B.4) and $\hat{v}(U_i, \mathbf{Y})$ defined in the proof of Theorem 1. One can show that $\frac{\partial}{\partial u}[\bar{v}(u, \mathbf{y}) - \hat{v}(u, \mathbf{y})] < 0$, implying (by Lemma 1) that

$$\hat{w}_e^{n-2}(x, y, \mathbf{y}_{-1}, y) - E[\max\{U_i, Z_2\} | X_i = x, \mathbf{Y} = \{y, \mathbf{y}_{-1}\}]$$

is decreasing in x . Thus, a necessary and sufficient condition for (D.4) to hold for all $x \geq y$ is

$$b^{n-2}(y, \mathbf{y}_{-1}) \geq \hat{v}_e^{n-2}(y, y, \mathbf{y}_{-1}, y) - \{ \hat{w}_e^{n-2}(1, y, \mathbf{y}_{-1}, y) - E[\max\{U_i, Z_2\} | X_i = 1, \mathbf{Y} = \{y, \mathbf{y}_{-1}\}] \} \quad \forall y, \mathbf{y}_{-1}. \tag{i.b}$$

This condition also ensures that even a winner with type $x < y$ (off the equilibrium path) prefers to make the pooling bid in phase $n - 1$. Hence, if condition (i) in Theorem 9 is replaced by

$$(i.a) \quad \tilde{b}_{e,e}^{n-1}(x; \mathbf{y}) = \hat{v}_e^{n-2}(y_1, y_1, \mathbf{y}_{-1}, y_1) \quad \forall x \geq y_1, \mathbf{y}_{-1}$$

and (i.b) above, we obtain a characterization incorporating these additional equilibria.

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