Bubbles and Crashes*

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Abstract

We present a model in which an asset bubble can persist despite the presence of rational arbitrageurs. The resilience of the bubble stems from the inability of arbitrageurs to temporarily coordinate their selling strategies. This *synchronization problem* together with the individual incentive to *time the market* results in the persistence of bubbles over a substantial period of time. The model provides a natural setting in which public events, by enabling synchronization, can have a disproportionate impact relative to their intrinsic informational content.

Keywords: Bubbles, Crashes, Temporal Coordination, Synchronization, Market Timing, 'Overreaction', Limits of Arbitrage, Behavioral Finance

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1 Introduction

This paper investigates the ability of an asset bubble to survive in the presence of rational arbitrage. Our conclusions suggest that arbitrage 'ultimately' works, though it might be ineffectual over substantial periods. In our setting, a bubble survives even though rational agents know that the bubble must burst with probability one in finite time.

Imagine a world in which there are *some* 'behavioral' agents variously subject to animal spirits, fads and fashions, overconfidence and related psychological biases which might lead to momentum trading, trend chasing and the like. There is by now a large literature which documents and models such behavior.[1] We do not investigate why behavioral biases needing rational corrections arise in the first place; for this we rely on the body of work very partially documented in footnote 1. The classical view is that arbitrage corrects such mispricing. Our concern in this paper is the extent to which such departures from efficient pricing can *persist* despite the presence of rational arbitrageurs. Thus our model provides further support for a variety of behavioral finance models of mispricing, many of which do not explicitly incorporate optimizing traders.

Suppose rational arbitrageurs understand that eventually the market will collapse but meanwhile would like to ride the bubble as it continues to grow.[2] Ideally, they would like to exit the market just prior to the crash. However, market timing is a difficult task. Our investors understand that they will, for a variety of reasons, come up with different solutions to this optimal timing problem. This dispersion of exit strategies and the consequent lack of synchronization are precisely what permit the bubble to thrive and grow. Selling pressure only impacts the bubble when a sufficient mass of traders sells out. All this is consistent with a world in which temporally coordinated selling by professional investors would lead to an immediate collapse of the bubble. This scenario is the starting point for our work.

We present a model which formalizes the above synchronization problem and yields a new perspective on the existence, persistence, and collapse of bubbles. We assume that the price surpasses the fundamental value at a random point in time t_0. Thereafter, arbitrageurs become sequentially aware that the price exceeds the fundamental value. After all rational arbitrageurs become aware of this mispricing, we consider a bubble to have emerged. Since arbitrageurs become sequentially aware of the mispricing, their trading strategies are initialized at different random times after t_0.[3] In

[1] Cognitive biases are illustrated in books and articles, such as Daniel, Hirshleifer, and Subrahmanyam (1998), Hirshleifer (2001), Odean (1998), Thaler (1991), Shiller (2000), and Shleifer (2000).

[2] The trends in internet stocks during the year 2000 can be viewed in this light. Most of the prominent institutional traders knew that the stock price of many internet companies could not be justified by their fundamentals. Nevertheless, only a few investors acted accordingly, while the majority could not afford to stay out of this fast growing market.

[3] Our theoretical framework is related to the working paper by Morris (1995) which we became

addition to its literal interpretation this assumption can be viewed as a metaphor for a variety of factors such as differences of opinion and information which, in particular, find expression in the factor we seek to explore and emphasize, that is, temporal miscoordination. Agents understand the structure of the model and in the simplest setting believe that it is equally likely that they became 'aware' of the bubble before or after the median agent.

Arbitrageurs can 'attack' the bubble at any point in time. They can sell their stock holding and can even go short in stocks. However, arbitrageurs face financial constraints which limit their stock holding as well as their maximum short-position.[4] This limits the price impact of each arbitrageur. Large price movements can only occur if sufficiently many traders sell or buy at the same time. We model this effect by assuming that the bubble only bursts when the selling pressure exceeds some threshold κ. In other words, a permanent shift in price levels requires a coordinated attack. In this respect, our model shares some features with the static second generation models of currency attack in the international finance literature (Obstfeld 1996).

However, these currency attack models focus exclusively on the question of *whether* to attack or not but ignore the important *temporal* aspect of coordination. Coordinating on a given action is complicated by the need to coordinate both the action and the time at which it is taken. Thus, speculators need to decide both whether or not to attack - the problem that is traditionally emphasized in the currency attack literature - and also 'when' to attack. It would be futile to simply coordinate on the 'whether' if it were not possible to also coordinate on the 'when'. This temporal element exacerbates the underlying coordination problem.

Notice that our model has elements of both cooperation and competition. Bursting the bubble requires some coordination among arbitrageurs, while arbitrageurs are also competitive in the sense that nobody wants to be too late to leave the market.

Our model generates several intriguing results. In our setting rational traders know that the bubble will burst for exogenous reasons by some time $t_0 + \bar{\tau}$ if it has not succumbed to endogenous selling pressure prior to that time. Here t_0 is the unknown time at which price path surpasses the fundamental value and arbitrageurs start getting aware of the bubble. Our analysis shows that if arbitrageurs' opinion is sufficiently

aware of after the first draft or our paper was written. Morris' paper analyses a standard dynamic coordination game using a model with "asynchronous clocks." His paper, in turn attributes the idea of "asynchronous clocks" to the computer science literature, in particular, to Halpern and Moses (1990). We will discuss his work further in Section 2.

[4]There is a growing literature which justifies restrictions on the possible portfolio positions of arbitrageurs. Shleifer and Vishny (1997) argue that insitutional investors, who invest on behalf of others, do not fully exploit long-run arbitrage opportunities out of fear of fund withdrawal. These draw downs might cause early liquidation before the arbitrage strategy pays off. Another example is margin requirements. They limit the degree of short sales. Borrowing constraints and the size of internal funds restrict the arbitrageur's stock holding position. Rather than explicitly modeling these financial constraints, we exogenously assume a restricted maximum portfolio position.

dispersed, there exists an equilibrium in which the bubble never bursts prior to $t_0 + \bar{\tau}$. Even long after the bubble begins and after all agents are aware of the bubble ($\bar{\tau}$ may be arbitrarily large), it is nevertheless the case that endogenous selling pressure is never high enough to burst the bubble. Moreover, this equilibrium is *unique*.[5] Thus there is a striking failure of the backwards induction argument which would yield immediate collapse in a standard model. The persistence of the bubble in our model relies on dispersion in traders' viewpoints about when the bubble emerged. Presumably, this dispersion is specially high at times of significant technological changes, such as the invention of the steam boat, telegraph, internet, etc. Structural breaks such as large scale financial liberalization programs can be another breeding ground for bubbles, as illustrated by the economic developments in the Scandinavian countries during the late eighties.

Second, we show that while arbitrageurs never burst a bubble if their opinions are sufficiently dispersed or if the absorption capacity by the behavioral momentum traders κ is very large, for smaller dispersion of opinion or for smaller κ, endogenous selling pressure advances the date at which the bubble eventually collapses. Nevertheless, the bubble grows for a substantial period of time.

The model also provides a natural setting in which news events can have a disproportionate impact relative to their intrinsic information content. This is because news events make it possible for agents to synchronize their exit strategies. Of course, large price drops are themselves significant public events, and we investigate how an initial price drop may lead to a full-fledged collapse. Thus the model yields a rudimentary theory of 'overreaction' and price 'cascades' and suggests a rationale for psychological benchmarks such as 'resistance lines'. In addition, our model provides a framework for understanding fads in information such as the (over-)emphasis on trade figures in the eighties and on interest rates in the nineties. Finally, our model supports arguments in favor of centralized news dissemination since news which is received sequentially over a long interval is much less likely to be reflected in the price.

The remainder of the paper is organized as follows. Section 2 illustrates how the analysis relates to the literature. In Section 3 we introduce the primitives of the model and define the equilibrium. Section 4 analyzes the case of a finite bubble which bursts after $\bar{\tau}$ periods for exogenous reasons even in the absence of any arbitrage. It illustrates why a lack of common knowledge leads to a failure of backwards induction. Section 5 analyses the case where the crash is caused, with a positive probability, by endogenous selling pressure from arbitrageurs. Section 6 highlights the special role of public events and discusses the fragility of bubbles with respect to different forms of public events. Section 7 concludes the analysis. Detailed proofs are relegated to the Appendix.

[5] We restrict attention to perfect Bayesian equilibria in which when an agent attacks, all agents who become aware of the bubble prior to the agent in question, also attack. We view this "monotonicity" restriction as both natural and innocuous in the context of the issues we seek to investigate.

2 Related Literature

We are not the first to note that arbitrage may be limited. The existing literature on limited arbitrage offers various justifications for our assumption that arbitrageurs are financially constrained, while also providing alternative rationales for mispricing different from the one developed here. In De Long, Shleifer, Summers, and Waldmann (1990) risk averse arbitrageurs do not correct the price because they are short-lived and thus only worry about the next period's noisy price instead of the riskless long-run fundamental value. In Shleifer and Vishny (1997), fund managers limit their arbitrage out of fear of a drawdown. Fund managers are afraid that their investors will withdraw their money if they suffer intermediate short-term losses even though the arbitrage provides a riskless payoff in the long run. These papers build on the insight that distorted prices might become even more distorted in the short run before eventually returning to their normal long run values. The logic of our model is rather different. In our model, arbitrage is limited due to the synchronization problem and the individual incentive to time the market. Synchronization problems not only provide an alternative reason for persistent mispricing but also amplify the effects described in the earlier literature on limited arbitrage.

Our work is also related to different branches of the literature on bubbles and mispricing in general.[6] The classical asset pricing literature typically rules out the existence of bubbles. The basic argument entails backwards induction from the last possible date at which the bubble can survive, given exogenous constraints. A comprehensive treatment of the non-existence of bubbles under symmetric information is provided in Santos and Woodford (1997).[7]

The prior literature on crashes follows approaches that are quite distinct from ours. We briefly note some of the key contributions. In Grossman (1988) a temporary crash occurs because professional market timers do not provide sufficient liquidity since they underestimate the degree to which other market participants' risk aversion increases as their wealth declines. More specifically, they underestimate other traders' portfolio insurance trading which synthesizes call-option payoffs. Gennotte and Leland (1990) develop a static setting in which the unknown extent of portfolio insurance trading leads to an 'inverted' S-shaped aggregate demand curve and thus to multiple equilibria. A small shift in the aggregate supply can lead to a discrete price drop. In Romer (1993)'s two-period rational expectations model, a small commonly observed supply shift can lead to a large price change. Our paper shares the feature that public news can have

[6] An extensive suvey of these models can be found in Brunnermeier (2001).

[7] Some other models of bubbles, starting with Tirole (1982), entail asymmetric information about fundamentals. Allen, Morris, and Postlewaite (1993) and Morris, Postlewaite, and Shin (1995) introduce higher order asymmetric information about the true value. The principal-agent structure in Allen and Gorton (1993) induces portfolio managers to churn bubbles. Even though higher order uncertainty is a defining feature of our setting, this uncertainty relates primarily to the timing rather than the fundamental value.

a disproportionate impact. In our setting, even a public event with no informational content can lead to the bursting of a bubble if the news arrival was unexpected. In addition to the overreaction to news events, we are also able to generate price cascades.

Our modeling approach is more closely related to the international finance currency attack literature. As in Obstfeld (1996) a price correction only occurs if there is sufficient selling pressure. Morris and Shin (1998) eliminate the indeterminacy of second generation speculative attack models by using the "global games approach" originally proposed by Carlsson and van Damme (1993). Our game is also a global game in the general sense of Carlsson and van Damme (1993) who define a global game to be "... an incomplete information game where the actual payoff structure is determined by a random draw from a given class of games and where each player makes a noisy observation of the selected game."[8] The results for static global games do not apply to our dynamic setting because learning from the existence of the bubble destroys strategic complementarity. See footnote 14 for a discussion of this point.

Our work is related to the notion of "asynchronous clocks" which appears in the computer science literature (see Halpern and Moses (1990)). Morris (1995) uses this framework to show that in a repeated coordination game players never achieve the efficient equilibrium, if they cannot perfectly synchronize their actions. The details of our respective applications and the structure of the analysis are significantly different. In particular, in our work asynchronicity is partially derived (our agents have access to synchronized clocks). Furthermore, Morris' (1995) analysis is closely related to the global games approach, whereas our model lacks the strategic complementarity upon which this approach relies.

Finally we note that some of the key elements of our model echothemes from Keynes (1936). The connections are elaborated upon in the discussion following the presenta-

[8] All the known results for global games entail substantial additional assumptions. The seminal paper by Carlsson and van Damme (1993) presents results for two player, two action games. These results have been adapted and extensively applied in a variety of economic settings involving coordination problems by Morris and Shin (1999), (2000), and others following up on their work, such as Goldstein and Pauzner (2000). More recently, Frankel, Morris, and Pauzner (2000) extend the results of Carlsson and van Damme (1993) to the context of more general games with strategic complementarities. We are not aware of any known results for global games which apply to our model. In addition to the restrictions noted above, their results are for static games; our model is dynamic. Furthermore, they typically let the noise of the signals going to zero. In our model, it is important that signals are sufficiently dispersed; indeed arbitraguers prick the bubble if dispersion is below a certain threshold level. Finally we note the two papers Burdzy, Frankel, and Pauzner (2001) and Frankel and Pauzner (1998) which seek to eliminate the multiplicity of equilibria in dynamic coordination failure games, the latter within a the specific framework of Matsuyama's (1991) dynamic two sector model. A key feature of these models is that agents are subject to frictions; in particular they can only change their actions at random times. This inertia, together with other assumptions regarding permanent payoff shocks generate unique equilibrium behavior often by iterative dominance arguments. Asymmetric information typically does not play a role in this body of work. Our own work relies on asymmetric information; however, the asymmetric information primarily concerns the temporal structure of the model and leads to a synchronization problem.

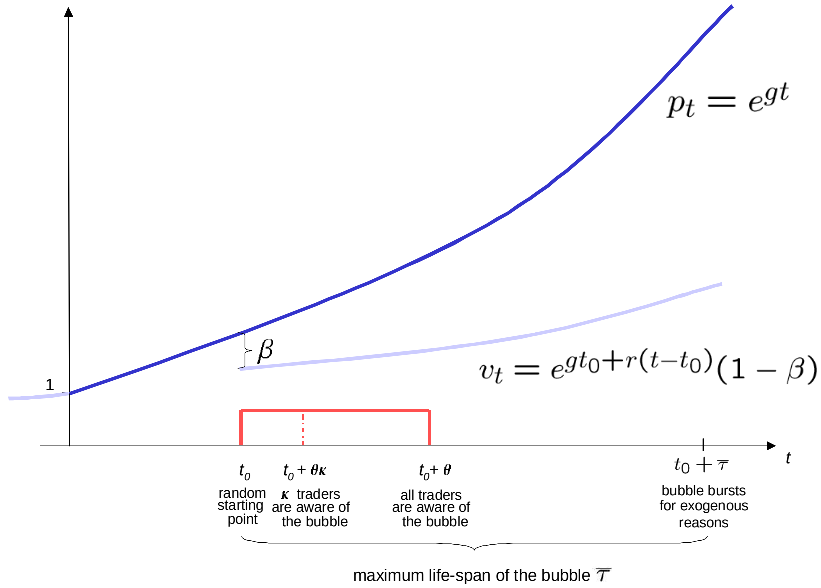

Figure 1:

tion of the model setup.

3 Model

3.1 Model Step

We wish to develop a model in which rational arbitrageurs become sequentially aware of mispricing starting at some random time t_0. For simplicity, we assume the following exogenous price process, depicted in Figure 1.

This price process may be motivated as follows. Prior to $t = 0$ the stock price index coincides with its fundamental value which grows at the risk-free interest rate r and rational arbitrageurs are fully invested in the stock market. Without loss of generality, we normalize the stock market price at $t = 0$ to $p_0 = 1$. From $t = 0$ onwards, both the fundamental value v_t as well as the stock price p_t grow at a rate of $g > r$, that is $p_t = e^{gt}$. This higher growth rate may be viewed as emerging from a series of unusual positive shocks which gradually make investors more and more optimistic about future prospects. Such shocks may be associated with new technologies, dramatic institutional changes such as financial or trade liberalizations, and so on. From some random time t_0, rational arbitrageurs become sequentially aware that the price is too high. At t_0 the "fundamental value" drops to $(1-\beta)\,e^{gt_0}$ (with $\beta \in [0,1]$) and thereafter grows at the "old economy" rate r. We assume that t_0 is exponentially distributed on $[0,\infty)$

with distribution function $F(t_0) = 1 - e^{-\lambda t_0}$. After t_0, behavioral momentum traders continue to keep the price growing at rate g, unless a mass of at least κ arbitrageurs sell their stock holding. Any selling pressure below a certain threshold level κ may be viewed as usual day-to-day fluctuations in order-flow which is absorbed by behavioral momentum traders. Even if the selling pressure never exceeds κ we assume that the bubble bursts for exogenous reasons at $t_0 + \bar{\tau}$. Note that this assumption of a final date is arguably the least conducive to the persistence of bubbles. In a classical model it would lead to an immediate collapse for the usual backwards programming reasons.

Another important element of our analysis is that rational arbitrageurs become sequentially informed that the fundamental value has not kept up with the growth of the stock price index. More specifically, a new cohort of rational arbitrageurs of mass $\frac{1}{\theta}$ becomes aware of the mispricing in each instant t from t_0 till $t_0 + \theta$. Since t_0 is random, an individual arbitrageur does not know whether other arbitrageurs have received the signal before or after them. Thus, the model contains two novel elements. First, agents become aware of the bubble sequentially. Second, agents do not know how many other rational arbitrageurs already know of the bubble's existence. Consequently, an agent who becomes aware of the bubble at t_i has a posterior distribution for t_0 with support $[t_i - \theta, t_i]$. Each agent views the market from the relative perspective of her own t_i. Viewed more abstractly, arbitrageurs' types are given by $\underline{t}_i \in [0, \infty)$, the date when they become aware of the bubble. Nature's choice of t_0 determines the 'active' types $\underline{t}_i \in [t_0, t_0 + \theta]$ in the economy. As noted earlier, we view this specification as a modeling device which captures temporal miscoordination arising from differences of opinion and information. The date t_i at which agent i becomes 'aware' of the mispricing may be more generally thought of as the date at which a player's strategy is 'initialized'.

Given such an environment, we want to identify the best strategy of a rational player $\underline{t}_i$. She knows t_i, the time when she first became aware of the mispricing. Let us denote the absolute time scale by t and the (relative) time elapsed since trader $\underline{t}_i$ became aware of the bubble by $\tau_i = t - t_i$. We will drop the subscript i, if τ_i is the same for all arbitrageurs.

After trader $\underline{t}_i$ becomes aware of the bubble, she can sell all or part of her stock holding or even go short till she reaches a certain limit where her financial constraint is binding. Each trader can also buy back shares. Without loss of generality, we can normalize the action space to be the continuum between $[0, 1]$, where 0 indicates the maximum long position and 1 the maximum short position each arbitrageur can take on. To avoid tedious technical qualifications, we assume that any change in a trader's portfolio holdings is maintained for some minimal period length $\Delta > 0$ and we characterize the limit equilibria as $\Delta \to 0$. Our model is therefore the continuous time limit of a more natural discrete time model. We will not invoke the Δ-qualification explicitly below, but it should be understood that it underlies our analysis.[9]

[9] Absent such qualifications our continuous time model would accommodate highly discontinuous strategies and best response would be unique only up to measure zero. Equivalently, we could also

The strategy of a trader who became aware of the bubble at time t_i is a function $\sigma_{\underline{t}_i} : [t_i, \infty) \mapsto [0,1]$, where $\left[1 - \sigma_{\underline{t}_i}(t)\right]$ can be viewed as trader $\underline{t}_i$'s current holding after she becomes aware of the bubble. A trader may exit from and return to the market multiple times. Given the structure of the game, the actions of the other traders affect trader $\underline{t}_i$'s payoff only if these actions cause the bubble to burst. Rather than specifying the payoff for each possible strategy, it suffices for our purposes to consider the payoff difference between selling (buying) $x_{t-\Delta}$ shares slightly earlier at $t - \Delta$ instead of at time t assuming that players adopt identical strategies before $t-\Delta$ and after t. Suppose $\left[1 - \sigma_{\underline{t}_i}(t - \Delta)\right] - x_{t-\Delta} \geq 0$. Selling $x_{t-\Delta}$ shares at $t-\Delta$, yields a revenue of $x_{t-\Delta}p_{t-\Delta}$, which invested in the money market account grows to $x_{t-\Delta}\left(e^{r\Delta}p_{t-\Delta}\right)$ at time t. Selling $x_{t-\Delta}$ shares at t, yields a revenue of $x_{t-\Delta}p_t$. Thus the difference between both 'cash payoffs' is given by $x_{t-\Delta}$ times

$$\left(e^{r\Delta}p_{t-\Delta} - p_t\right).$$

The actual prices depend on when the bubble bursts. If the bubble bursts within the interval $(t - \Delta, t)$, then the revenue difference is given by $x_{t-\Delta}$ times $e^{r\Delta}e^{g(t-\Delta)} - e^{gt_0 + r(t-t_0)}(1-\beta)$. Note that one also has to define the payoffs for the case where the bubble bursts exactly at the time when the arbitrageur submits her sell order, that is either at $t - \Delta$, or t. In this case all orders are submitted at the pre-crash price e^{gt} as long as the accumulated selling pressure is smaller than κ. If the accumulated selling pressure exceeds κ, then only the first randomly chosen orders will be executed at e^{gt} while the remaining orders are executed at the post-crash price $e^{gt_0 + r(t-t_0)}(1-\beta)$. In other words, the expected execution price

$$\alpha_t e^{gt} + (1-\alpha_t)\, e^{gt_0 + r(t-t_0)}(1-\beta)$$

is a convex combination of both prices. $\alpha < 1$ if the selling pressure is strictly larger than κ, and $\alpha = 1$ if the selling pressure is exactly κ at the time of the bursting of the bubble.

In addition, we introduce a reputational penalty for arbitrageurs who decide to (partially) stay out of the market in the case where the bubble does not burst. There are various rationales for this penalty. For example, institutional investors who are not fully invested in the market have a lower return on their portfolio and therefore their clients might withdraw their money. In this case they have to liquidate profitable long-run arbitrage opportunities. The reputational penalty leads to an additional difference in payoff of $cp_t\Delta x_{t-\Delta}$ if trader $\underline{t}_i$ sells $x_{t-\Delta}$ shares at $t-\Delta$ instead of at t and the bubble does not burst. That is, we assume that the penalty is proportional to the stock market index p_t, the time length Δ and the number of shares $x_{t-\Delta}$ she sells early.[10]

assume that there are ε-costs per unit trade and focus on the case where ε goes to zero.

[10] Tony Dye's case provides a vivid illustration of this penalty. He was for many years the successful

3.2 Discussion of the Model

We now discuss and put into perspective some key elements of our model. The fundamental question whether professional arbitrageurs correct mispricings is a very old one and goes back to at least Keynes' "General Theory of Employment, Interest and Money" (1936) in which he wrote:

> It might have been supposed that *competition between expert professionals*, possessing judgment and knowledge beyond that of the average private investor, *would correct* the vagaries of the ignorant individual left to himself. *(italics added)*

Our model setup relies crucially on two elements, which can also be traced back to Keynes (1936). First, professional arbitrageurs want to ride the bubble as long as possible given that the bubble grows at a rate g larger than the riskfree rate r. Each individual trader's objective is to leave the market just prior to the crash. Professional traders do not want to forego the capital gains as long as the bubble grows, but, of course, seek to exit before the crash. Referring back to Keynes:

> The actual, *private* object of the most skilled investment to-day is *"to beat the gun"*, as the Americans so well express it, to outwit the crowd, and to pass the bad, or depreciating, half-crown to the other fellow. *(italics added)*

The second element is that large price changes require a certain degree of coordination. In our model the bubble only bursts if more than κ traders attack the bubble at the same time. Therefore, each professional arbitrageur only attacks at times when she believes that the other arbitrageurs attack as well. Hence, it is more important to focus on other arbitrageurs' trading than on the fundamentals of an asset. This is in the same spirit as "Keynes' Beauty Contest":[11]

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chief investment officier of Phillips and Drew, London. Nicknamed "Dr. Doom" he refused to bow to the fashion of investing in internet stocks fearing an imminent slump in the markets. He lost his job in March 2000 - just days before his warnings that the tech bubble would burst came true. In market historian David Schwartz's words "The irony is he [Tony Dye] may well be right, but at the wrong time." Even though this penalty is very relevant in today's financial markets, our analysis does not rely on it. All results also hold for $c = 0$.

[11] Other authors also have found inspiration in this famous quote in Keynes (1936). Froot, Scharfstein, and Stein (1992) focus on the endogenous information acquisition decision of traders. Their analysis relies on the fact that traders are short-lived and might be forced to unwind their position before the collected information is reflected in the price. Consequently, traders have an incentive to collect information which other traders will also collect. In contrast, we focus on the persistence of bubbles in the presence of rational arbitrageurs without restricting their horizons.

> ... professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, *all of whom are looking at the problem from the same point of view. (italics added)*

Indeed, in our relativistic framework, each trader looks at the problem from the same point of view, however, *relative to the date*, when she became aware of the bubble's existence. Combining these two elements with our earlier assumption that traders become aware of the bubble in a sequential random order leads to our results.

In our model, we have exogenously assumed that the bubble only bursts when the selling pressure exceeds κ. In general, we think that small selling pressure has a negligible price impact, in particular if this selling pressure is of a temporary nature. Market makers and risk-neutral day traders[12] view temporary selling pressure as a cheap buying opportunity and make excess profits as long as the bubble continues to grow. Behavioral momentum traders may absorb any permanent selling pressure originating from rational arbitrageurs' trading activity. We have not modelled the behavior of boundedly rational traders. This leaves many modelling alternatives open and also allows for multiple interpretations of κ. For example, we could introduce wealth-constrained risk-neutral momentum traders in our model. Imagine a situation where the momentum traders' estimate of the value of the stock increases exponentially at a rate of g. Since they are risk-neutral, they determine the price process as long as they can absorb the shares sold by the arbitrageurs. When the rational arbitrageurs' selling pressure surpasses κ, they are unable to take on larger positions and realize their misperception. In this setup, κ has a nice interpretation since it reflects the mass of momentum traders.

One potential criticism is that by assuming that the bubble only bursts when the selling pressure exceeds κ, we implicitly also assume that rational agents do not become aware of selling pressure by other rational agents until it crosses a certain threshold. Our response is that we view the model as being a stylized, one-dimensional, depiction of a far more complex setting with noisy prices and agents differing in the manner in which they interpret noisy signals. This leads them to have different perceptions of the probability with which the bubble is likely to burst now and in the future. Initially, the threshold at which agents become aware of the selling pressure is taken to be the point at which the bubble actually bursts. We relax this strong assumption in Section 6. This permits us to analyze price cascades. Smaller selling pressure might already

[12] Day traders do not engage in fundamental analysis. They solely rely on information inferred from past and current prices.

lead to a minor price decline, which in turn might be used as a synchronization device. Given the right circumstances, this might develop into a full blown crash.

3.3 A Preliminary Result - Attack Condition

Let $\Pi(t|\underline{t}_i)$ denote trader $\underline{t}_i$'s unconditional cumulative distribution function that the bubble bursts prior to time t. Note that $\Pi(t|\underline{t}_i)$ of trader $\underline{t}_i$ depends on the specifics of the model structure as well as on the equilibrium conjecture by trader $\underline{t}_i$. We suppress the latter arguments for simplicity. Therefore, the expected payoff difference from selling one share at $t-\Delta$ instead of at t, is given by

$$\left(\frac{\Pi(t|t_i)-\Pi(t-\Delta|t_i)}{1-\Pi(t-\Delta|t_i)}\right)\left\{\alpha_{t-\Delta}e^{g(t-\Delta)+r\Delta}-\alpha_{t-\Delta}e^{gt_0+r(t-t_0)}(1-\beta)\right\}+$$
$$+\left(1-\frac{\Pi(t|t_i)-\Pi(t-\Delta|t_i)}{1-\Pi(t-\Delta|t_i)}\right)\left\{e^{r\Delta}e^{g(t-\Delta)}-\left[\alpha_t e^{gt}+(1-\alpha_t)\,e^{gt_0+r(t-t_0)}(1-\beta)\right]\right\}.$$

Suppose $\Pi(t|\underline{t}_i)$ admits density $\pi(t|\underline{t}_i)$, that is, $\Pi(t|\underline{t}_i)=\int_{-\infty}^{t}\pi(t'|\underline{t}_i)\,dt'$. Let $\pi^c(t|\underline{t}_i)=\frac{\pi(t|\underline{t}_i)}{1-\Pi(t|\underline{t}_i)}$ denote the corresponding conditional density, given that the bubble still exists at t.

Lemma 1 ***(Attack condition)*** *Suppose $\Pi(t|\underline{t}_i)$ admits density $\pi(t|\underline{t}_i)$. Then if*

$$\pi^c(t|\underline{t}_i)<\frac{g-r+c}{1-(1-\beta)\,E\left[e^{-(g-r)(t-t_0)}|\underline{t}_i,t\right]}$$

trader $\underline{t}_i$ will choose to trade to the maximum long position. Conversely, if $\pi^c(t|\underline{t}_i)>\frac{g-r+c}{1-(1-\beta)E\left[e^{-(g-r)(t-t_0)}|\underline{t}_i,t\right]}$ she will trade to the maximum short position.

Proof. In the case where the density $\pi_i(t|\underline{t}_i)$ is well defined, the bubble bursts in each instant with zero probability. Therefore, we can ignore the zero-probability cases where the selling pressure strictly exceeds κ exactly at $t-\Delta$ or t and leads to a bursting of the bubble: We can, without loss of generality, set $\alpha_{t-\Delta}=\alpha_t=1$.
The payoff difference between selling $x_{t-\Delta}$ shares at $t-\Delta$ instead of t, is then given by $x_{t-\Delta}$ times

$$\left(\int_{t-\Delta}^{t}\pi^c(t'|\underline{t}_i)\,dt'\right)\left\{\left[e^{gt}-(1-\beta)\,E\left[e^{(g-r)t_0}|\underline{t}_i,t\right]e^{rt}\right](1+\Delta c)\right\}+$$
$$+\left\{\left(e^{g(t-\Delta)}e^{r\Delta}-e^{gt}\right)-\Delta c e^{gt}\right\}$$

Using a first order Taylor expansion around t, (i.e. $\Delta=0$) the expected payoff difference simplifies to $x_{t-\Delta}e^{gt}$ times

$$\Delta\pi^c(t|\underline{t}_i)\left[1-(1-\beta)\,E\left[e^{(g-r)t_0}|\underline{t}_i,t\right]e^{-(g-r)t}\right](1+\Delta c)-(g-r)-c.$$

For $\Delta \rightarrow 0$ we can ignore terms of order $(\Delta)^2$ and hence trader $\underline{t}_i$ should sell $x_{t-\Delta}$ shares already at t if

$$\pi^c (t|\underline{t}_i) > \frac{(g-r)+c}{1-(1-\beta)\, E\left[e^{-(g-r)(t-t_0)}|\underline{t}_i, t\right]} \tag{*}$$

and hold them at t, when the inequality is reversed. Since selling (holding) $x_{t-\Delta}$ shares affects the payoff linearly, it is optimal to go to the maximum short (long) position. ■

The attack condition (*) is easier to interpret if we introduce Δ-terms of higher order.

$$\Delta\pi^c (t|t_i) \left[e^{gt} - (1-\beta)\, E\left[e^{(g-r)t_0}|t_i, t\right]\right] > (1 - \Delta\pi^c (t|t_i)) \left[(g-r)+c\right] e^{gt}\Delta$$

The left hand side reflects the expected benefits of being out of the stock market. The bubble of estimated size $\left[e^{gt} - (1-\beta)\, E\left[e^{(g-r)t_0}|t_i, t\right]\right]$ bursts with a probability $\Delta\pi^c (t|t_i)$ during the small interval Δ. The right hand side captures the costs of being out of the market for a short interval Δ in the case that the bubble does not burst. Not only does the arbitrageur lose out on the appreciation $(g-r)$, the arbitrageur also suffers a reputational penalty c. The lemma also asserts that trader $\underline{t}_i$ either wholeheartedly attacks or holds the maximum long position. Effectively, the relevant action space reduces to $\{0, 1\}$. The arbitrageur's risk neutrality enables us to take this simplifying step.

Before proceeding to the analysis of specific models, let us define the equilibrium concept and discuss the underlying assumptions of the model.

Definition 1 *A* ***trading equilibrium*** *is defined as a Perfect Bayesian Nash Equilibrium in which a trader who attacks at time t (correctly) believes that all traders who became aware of the bubble prior to her also attack at t.*

This restriction on beliefs is very natural in our setting since in all variations of the model that we consider, it becomes easier to successfully attack the bubble the longer it persists. Note that we are **not** restricting attention to trigger strategies in which an agent who attacks at t continues to attack at all times thereafter.

4 Persistence of Bubbles

4.1 Analysis

To contrast our analysis with the standard backwards induction outcome, we assume that in the absence of rational arbitrage the bubble bursts precisely $\bar{\tau}$ periods after the mispricing emerged. That is, the bubble bursts at $\bar{t} = t_0 + \bar{\tau}$ for exogenous reasons in the event that selling pressure prior to $t_0 + \bar{\tau}$ never exceeds the threshold κ. This fact is common knowledge. In a model in which all arbitrageurs became aware of the bubble at t_0, all traders would attack the bubble and it would burst immediately. This is a consequence of the usual backwards induction argument. In our model, arbitrageurs become aware of the bubble sequentially in a random order and furthermore have a non-degenerate posterior distribution over t_0. Recall that traders become 'aware' of the bubble during the interval $[t_0, t_0 + \theta]$, where we have interpreted θ to be a measure of differences in opinion and other heterogeneities across players. Following the terminology used in Allen, Morris, and Postlewaite (1993), the mispricing becomes a bubble after it is known to all arbitrageurs, that is at $t_0 + \theta$.

We show that if θ is not too small, that is, if $\theta > \frac{1}{\kappa}\frac{-\ln\left(1-\frac{\lambda}{g-r+c}\right)}{\lambda}$, then the trading equilibrium is unique and the bubble bursts precisely at $t_0+\bar{\tau}$ for all t_0. In this case the endogenous selling pressure of the rational arbitrageurs has absolutely no influence on the time at which the bubble bursts. It is worth noting that this result holds despite the fact that it is possible within our model for traders to coordinate on particular dates, say Friday, 13th of April 2001, by adopting strategies which are not symmetric in τ_i.

Proposition 1 *Suppose* $\theta\kappa > -\ln\left(1-\frac{\lambda}{g-r+c}\right)\left(\frac{1}{\lambda}\right)$, $g-r+c > \lambda$. *Then there exists a unique trading equilibrium. In this equilibrium all traders begin attacking* $\tau^* = \bar{\tau} - \left(\frac{1}{\lambda}\right)\left[-\ln\left(1-\frac{\lambda\left(1-(1-\beta)e^{-(g-r)\bar{\tau}}\right)}{g-r+c}\right)\right] < \bar{\tau}$ *periods after they became aware of the bubble and continue attacking thereafter. Nevertheless, for all* t_0, *the bubble bursts precisely at* $t_0 + \bar{\tau}$.

For a given trading equilibrium, we define the function ψ which specifies for each t the minimum τ_i at which some arbitrageur attacks at t. Given our definition of a trading equilibrium any trader $\underline{t}_{i'}$ with $\tau_{i'} \geq \psi(t)$ also attacks at t, and a trading equilibrium is completely characterized by its associated ψ-function. Indeed, we will sometimes refer to an equilibrium ψ. For a given t_0 the total mass of traders attacking at $t \geq t_0$ is

$$\begin{aligned} s_{t_0,t} &\equiv \min\left\{\int_{t_0}^{t-\psi(t)} \frac{1}{\theta} dt', 1\right\} \\ &= \min\left\{\frac{1}{\theta}\left(t-\psi(t)-t_0\right), 1\right\}. \end{aligned}$$

Lemma 1 shows that all traders always either hold their maximum or their minimum position. That is, in equilibrium $\sigma_{\underline{t}_i}$ must equal 0 or 1, respectively. Consequently, $s_{t_0,t}$ also corresponds to the aggregate selling pressure.

The following two Lemmas are useful in establishing Proposition 1. We focus on the attacking traders $t_i = t - \psi(t)$, whose time lag between awareness and action is shortest at a given time t.

Lemma 2 ***(Preemption Lemma)*** *Consider for each t a trader $\underline{t}_i = t - \psi(t)$ who first attacks at time $t > t_i$. In equilibrium trader $\underline{t}_i$ cannot believe with strictly positive probability that $s_{t_0,t} > \kappa$.*

Proof. Suppose trader $\underline{t}_i$ attacks the first time t when $\Pr_i(t_0 | s_{t,t_0} > \kappa) > 0$. That is, with strictly positive probability the bubble will burst and since s_{t,t_0} strictly exceeds κ, the selling price is already reduced by a fraction $\alpha < 1$. Consequently, there exists a small $\Delta > 0$ such that attacking at $t - \Delta$ leads to a strictly larger expected payoff for trader $\underline{t}_i$. ■

The motive to pre-empt a possible crash rules out equilibria in which a mass of 'aware' arbitrageurs start attacking on a specific date, say Friday, April 13th of 2001.

Lemma 2, together with the fact that each attacking trader believes that traders who became aware of the bubble before her also attack, allows us to derive a necessary condition for the equilibrium beliefs of arbitrageur $\underline{t}_j$ whose time lag between awareness and attacking is shortest. Formally, trader $\underline{t}_j$'s $\tau_j := \min_t \psi(t)$. The formal proof is relegated to Appendix A.1.

Lemma 3 *Suppose arbitrageur $\underline{t}_j$ first attacks at some $t > t_j$, then it must be the case that player $\underline{t}_j$ believes that the support of t_0 is $[\max\{t_j - \theta\kappa, t - \bar{\tau}\}, t_j]$.*

Equipped with these two lemmas we provide the proof of Proposition 1 in the Appendix A.2. The intuition for this result is probably best provided by a less general but more intuitive argument.

Suppose that for a given strategy profile, endogenous selling pressure would eventually burst the bubble at some $t_0 + \hat{\tau}$, where $\hat{\tau} < \bar{\tau}$. Then $\hat{\tau}$ is mutual knowledge among all arbitrageurs, while t_0, of course, is not known. At each date t, each arbitrageur $\underline{t}_i$ tries to predict the likelihood that the bubble might burst in the next instant. For $t < t_i - \theta + \hat{\tau}$, trader $\underline{t}_i$ can rule out the possibility that the bubble bursts in the next instant. For $t \geq t_i - \theta + \hat{\tau}$, trader $\underline{t}_i$ believes that the date $t_0 + \hat{\tau}$ at which the bubble bursts is uniformly distributed between the next instant t and $t_i + \hat{\tau}$. The latter is the case if trader $\underline{t}_i$ became aware of the bubble precisely at t_0. That is, the conditional density that the bubble might burst at instant t is $\pi^c(t|\underline{t}_i) = \frac{\lambda}{1-e^{-\lambda(t_i+\hat{\tau}-t)}}$ for $t \geq t_i - \theta + \hat{\tau}$ and zero otherwise. The attack condition Lemma 1 states that trader $\underline{t}_i$ will only attack if $\pi^c(t|\underline{t}_i) = \frac{\lambda}{1-e^{-\lambda(t_i+\hat{\tau}-t)}} \geq \frac{(g-r)+c}{1-(1-\beta)E\left[e^{-(g-r)(t-t_0)}|\underline{t}_i,t\right]}$. Since the right hand side is always smaller than $\frac{(g-r)+c}{1-(1-\beta)e^{-(g-r)\bar{\tau}}}$, each arbitrageur $\underline{t}_i$ attacks earliest from

$$t \geq t_i + \hat{\tau} - \frac{-\ln\left(1 - \frac{\lambda\left(1-(1-\beta)e^{-(g-r)\bar{\tau}}\right)}{g-r+c}\right)}{\lambda}$$

onwards. Note that $\hat{\tau} \geq -\ln\left(1 - \frac{\lambda\left(1-(1-\beta)e^{-(g-r)\bar{\tau}}\right)}{g-r+c}\right)\left(\frac{1}{\lambda}\right)$, since no trader attacks before she is aware of the bubble. The mass of traders who are aware of the bubble at time t is given by $\max\left\{\frac{1}{\theta}(t-t_0), 1\right\}$. Given that each arbitrageur never attacks before $\hat{\tau}-\left(\frac{1}{\lambda}\right)\left[-\ln\left(1-\frac{\lambda\left(1-(1-\beta)e^{-(g-r)\bar{\tau}}\right)}{g-r+c}\right)\right]$ periods, the total selling pressure at time t is less than $\max\left\{\frac{1}{\theta}\left(t-\hat{\tau}+\left(\frac{1}{\lambda}\right)\left[-\ln\left(1-\frac{\lambda\left(1-(1-\beta)e^{-(g-r)\bar{\tau}}\right)}{g-r+c}\right)\right]-t_0\right), 1\right\}$. For $t=t_0+\hat{\tau}$ the total selling pressure is at most $\frac{1}{\theta}\left(\frac{1}{\lambda}\right)\left(-\ln\left(1-\frac{\lambda\left(1-(1-\beta)e^{-(g-r)\bar{\tau}}\right)}{g-r+c}\right)\right)$ which is smaller than the threshold κ. This contradicts the initial assumption that $\hat{\tau}<\bar{\tau}$. Therefore, we can rule out any equilibrium where $\hat{\tau}$ is independent of t_0. In particular, the proof shows that there does not exist a symmetric equilibrium in which the bubble bursts prior to $\bar{\tau}$.[13]

However, there might exist equilibria for which $\hat{\tau}$ is a function of t_0. For example, attacking on a particular date, say Friday, 13th of April 2001 (denoted by t^{13}) leads to

$$\hat{\tau}^{13}=\begin{cases}\hat{\tau} & \text{if } t_0>t^{13}-\theta\kappa \\ t^{13}-t_0 & \text{if } t_0\leq t^{13}-\theta\kappa\end{cases}$$

which is not ruled out as a possible equilibrium by the above argument. However, given this strategy profile, a positive mass of traders starts attacking at t^{13}. That is, the bubble bursts with a strictly positive probability from the individual trader's viewpoint. By Lemma 2 traders have an incentive to attack earlier and hence it is not an equilibrium strategy. The proof in the Appendix generalizes this result to any possible trading strategy profile.

At $t_0+\bar{\tau}$ the bubble bursts for exogenous reasons. In equilibrium each arbitrageur starts leaving the stock market at $t\geq t_i+\bar{\tau}-\left(\frac{1}{\lambda}\right)\left[-\ln\left(1-\frac{\lambda\left(1-(1-\beta)e^{-(g-r)\bar{\tau}}\right)}{g-r+c}\right)\right]$. Notice, however, that at the time of the crash $t_0+\bar{\tau}$, the endogenous selling pressure is less than κ and thus, the bubble bursts purely for exogenous reasons. This is in contrast to the standard backwards induction reasoning where the anticipation of a crash prepones the price drop.

We note here that the standard iterative dominance proof of global games cannot be applied in our setting, since our game does not exhibit strategic complementarities. The reason is that traders infer information from the fact that the bubble still exists.[14]

[13]Note that for $\lambda\to 0$, the prior distribution of t_0 is an (improper) uniform distribution. In this case the conditional distribution for trader $\underline{t}_i$ that the bubble bursts in the next instant is $\pi^c(t|\underline{t}_i)=\frac{1}{t_i+\bar{\tau}-t}$ and the term $-\ln\left(1-\frac{\lambda\left(1-(1-\beta)e^{-(g-r)\bar{\tau}}\right)}{g-r+c}\right)\left(\frac{1}{\lambda}\right)$ converges to $\frac{1-(1-\beta)e^{-(g-r)\bar{\tau}}}{(g-r)+c}$..

[14]This is probably best illustrated by means of an example, wherein we restrict the strategy space to trigger strategies. Consider a trader $\underline{t}_i$ who starts attacking the bubble at $t^{13}=t_i+\tau_i$, provided that all other traders attack immediately when they became aware of the bubble. Given this strategy profile, trader $\underline{t}_i$ can infer a lower bound for t_0 from the fact that the bubble still exists. Compare this with a situation where other traders do not start attacking immediately when they become aware

4.2 Lack of Common Knowledge

In standard models, any finite bubble can be eliminated by the classic backwards induction reasoning. However, the standard backwards induction argument requires a starting point which is common knowledge among the arbitrageurs. To gain a better understanding for why a bubble persists even though the life span of the bubble is finite, it is useful to take a closer look at arbitrageurs' knowledge of the bubble as time evolves. Proposition 2 below shows that it is never common knowledge that at least κ traders are aware of the bubble; this basic observation provides an alternative perspective on the difference between our model and the classical literature on bubbles.

Proposition 2 *It is never common knowledge that at least κ traders are aware of the bubble.*

Proof. It is sufficient to look at the first κ traders. At $t_0 + \theta\kappa$, at least κ traders know of the bubble. That is, it is mutual knowledge among κ traders at $t_0 + \theta\kappa$. However traders, in particular arbitrageurs who only became aware at $t_0 + \theta\kappa$, are not sure whether the other arbitrageurs are aware of the bubble too. At $t_0 + 2\theta\kappa$, the first κ traders know that a bubble exists and that at least a fraction κ of the arbitrageurs knows of the bubble. However, they do not know whether a fraction κ knows that a fraction κ knows that the bubble exists, etc. This is only the case at $t_0 + 3\theta\kappa$. More generally, let n be a positive integer, then at $t_0 + n\theta\kappa$, the κth trader knows that at least κ traders know that at least κ know that ... and so on at most n-times. It will never be common knowledge among κ traders that there are at least a fraction κ traders who know of the bubble. ■

As time goes to infinity the fact that at least κ arbitrageurs know of the bubble becomes "almost common knowledge" in the case where the bubble never bursts for exogenous reasons, that is $\bar{\tau} \to \infty$. However, it is never common knowledge among at least κ traders. Thus, our synchronization problem model is another example of the striking difference between common knowledge and almost common knowledge.[15]

5 Endogenous Crashes

The previous section highlights the central features of our modeling approach and emphasizes the different points of departure from traditional models. While the bubble bursts exactly after $\bar{\tau}$ periods for exogenous reasons in the previous section, we show in this section that rational arbitrageurs burst the bubble if arbitrageurs' dispersion of

of the bubble but only at, say t^{13}. In this case trader $\underline{t}_i$ cannot derive a lower bound for t_0 from the existence of the bubble. Consequently, she has a greater incentive to attack the bubble at t^{13}. This is exactly the opposite of what strategic complementarity would prescribe.

[15]This distinction was first introduced in the economics literature by Rubinstein (1989).

opinion and/or momentum traders' absorption capacity are sufficiently large. Hence, this analysis could be viewed as a bridge between the first part of our paper, where the bubble only bursts for exogenous reasons, and the standard 'no bubble' literature, where bubbles are ruled out since arbitrageurs would burst it right away.

We focus on the trading equilibrium which makes it hardest to sustain a bubble. That is, we focus on the equilibrium, where arbitrageurs attack earliest. This raises the bar for our analysis since our primary objective is to show that bubbles may exist despite the presence of rational arbitrageurs. Since each equilibrium can be characterized by its ψ-function, the more formal statement is that we focus on the equilibrium ψ^{**} with the property that $\psi^{**}(t) \leq \psi(t)$ for all t for any possible equilibrium function ψ. It is not obvious that a ψ^{**}-equilibrium exists. Proposition 3 below shows that it does, and furthermore that the "most aggressive" equilibrium ψ^{**} is a symmetric trigger-strategy equilibrium. The chosen setup also allows us to derive the endogenous life span of the bubble and the exact trading strategies in closed form.

Proposition 3 *Suppose* $\theta\kappa < \left(\frac{1}{\lambda}\right)\left[-\ln\left(1-\frac{\lambda-\lambda(1-\beta)e^{-(g-r)\bar{\tau}}}{g-r+c}\right)\right]$. *Then there exists a symmetric trigger strategy equilibrium, in which the traders leave the market* $\tau^{**} = \max\left\{0, -\ln\left(\frac{(g-r+c)e^{-\lambda\theta\kappa}-(g-r+c)+\lambda}{\lambda(1-\beta)}\right)\left(\frac{1}{g-r}\right)-\theta\kappa\right\}$ *periods after they become aware of the bubble. Hence, the bubble bursts exactly at*

$$t_0 + \max\left\{\theta\kappa, \frac{-\ln\left(\frac{\lambda-(g-r+c)\left(1-e^{-\lambda\theta\kappa}\right)}{\lambda(1-\beta)}\right)}{(g-r)}\right\}.$$

Furthermore $\tau^{**} \leq \psi(t)$ *for all* t *for any possible equilibrium* $\psi(t)$.

The proof of this proposition is presented in Appendix A.3. Note that for $\beta = 0$, $\theta\kappa \leq -\ln\left(\frac{\lambda-(g-r+c)\left(1-e^{-\lambda\theta\kappa}\right)}{\lambda(1-\beta)}\right)\left(\frac{1}{g-r}\right)$ and hence, the bubble always bursts at $t_0 - \ln\left(\frac{\lambda-(g-r+c)\left(1-e^{-\lambda\theta\kappa}\right)}{\lambda(1-\beta)}\right)\left(\frac{1}{g-r}\right)$.

The previous section highlights the point that the usual backwards induction argument breaks down since at no point is it common knowledge that a bubble exists. However, the above equilibrium can be obtained via the following (backward) procedure which entails "iterative removal of non-best-response symmetric trigger strategies." Since the bubble ultimately bursts for exogenous reasons, eventually attacking a bubble becomes a dominant strategy even when it is assumed that no other trader ever attacks. Let $\underline{t}_i + \tau^1$ $(= \underline{t}_i + \tau^*)$ be this date. Under the conjecture that each trader $\underline{t}_{i'}$ starts attacking from $\underline{t}_{i'} + \tau^1$ onwards, trader $\underline{t}_i$'s best (trigger) response is to attack even earlier from $\underline{t}_i + \tau^2$ onwards. Iterating this argument yields the symmetric trigger-strategy equilibrium described above. Intuitively, as τ_i approaches $\bar{\tau}$

arbitrageur $\underline{t}_i$ become sufficiently 'nervous' that the bubble might burst, which leads to an endogenous response. This endogenous response feeds on itself. Nevertheless, the bubble persists till $t_0 + \tau^{**} + \theta\kappa$.

The closed form solution enables us to conduct some comparative statics. The endogenous life-span of the bubble increases as the dispersion of opinions among arbitrageurs θ increases. Taking our model literally, θ describes the time span (window) over which traders become sequentially aware of the bubble. It is also essential for our argument to work that individual traders do not know when they became aware of the bubble relative to others: individual traders become aware of the bubble in a sequential, random order. The larger the window θ, the more uncertain is each arbitrageur about when other traders became aware of the bubble. Alternative model formulations show that the dispersion of the timing is crucial for the emergence of the bubble and not the difference in the estimate of the fundamental value. The comparative static of the absorption capacity κ of the momentum traders is the same as for θ. A larger κ requires more coordination among arbitrageurs and thus prolongs the bubble. As one would expect the reputational penalty c make it more costly to stay out of the market and, hence, extend the life-span of the bubble. A change in the excess growth rate of the bubble $(g - r)$ causes two opposing effects. On the one hand, it is more costly to not participate in the appreciation of the bubble. This effect is similar to the reputational penality c extending the larger life-span of the bubble. On the other hand, a higher (past) excess grows rate also increases the current size of the bubble. The larger the bubble, the more incentive the arbitrageur has to leave the stock market. Overall, the comparative static is ambivalent. For small $(g - r)$ an increase in the excess growth rate increases the life-span of the bubble, while the opposite is true for large $(g - r)$. Notice that a surprise interest rate increase by the Federal Reserve Bank might induce arbitrageurs to leave the stock market. It is not worth risking to be caught by a crash in exchange for a smaller excess return $(g - r)$. However, if the crash does not occur, the bubble grows at a lower rate which might lengthen the life-span of the bubble.

The proofs of Proposition 3 and Lemma 4 in the appendix allow us to replace the sufficient condition of Proposition 1 with a necessary and sufficient condition.

Proposition 4 *For a given $\bar{\tau}$, $\theta\kappa > \left(\frac{1}{\lambda}\right)\left[-\ln\left(1 - \frac{\lambda - \lambda(1-\beta)e^{-(g-r)\bar{\tau}}}{g-r+c}\right)\right]$ is the necessary and sufficient condition that the bubble bursts for exogenous reasons at $t_0 + \bar{\tau}$. Furthermore, this outcome is unique.*

Proof. It is easy to check that for $\theta\kappa = \left(\frac{1}{\lambda}\right)\left[-\ln\left(1 - \frac{\lambda - \lambda(1-\beta)e^{-(g-r)\bar{\tau}}}{g-r+c}\right)\right]$ the time when the bubble bursts for endogenous reasons $t_0 + \tau^{**} + \theta\kappa$ coincides with the time $t_0 + \bar{\tau}$, when it would burst for exogenous reasons. Since $\frac{\partial \tau^{**}}{\partial \theta\kappa} > 0$, the bubble bursts for exogenous reasons for $\theta\kappa > \left(\frac{1}{\lambda}\right)\left[-\ln\left(1 - \frac{\lambda - \lambda(1-\beta)e^{-(g-r)\bar{\tau}}}{g-r+c}\right)\right]$.

By Lemma 4 there exists no other equilibrium where the bubble bursts before $t_0 + \tau^{**} + \theta\kappa$ for endogenous reasons. Hence, the bursting at $t_0 + \bar{\tau}$ is unique outcome. ■

6 Synchronizing Public Events

For simplicity, we will restrict our formal analysis to sunspots which serve as pure coordination devices and will not consider signals which reveal information about t_0. This is consistent with our emphasis on issues of synchronization. Informative public signals would not entail qualitative changes to the analysis presented below.

Interestingly, *anticipated* public events do not alter the analysis in any way. To see this, observe that "Friday 13th of April 2001" is precisely such an anticipated public event and we have argued in the preceding section that an anticipated crash on a specific date cannot occur. The reason is that each individual trader has an incentive to pre-empt a proposed synchronized attack on April 13th. However, traders cannot front-run *unanticipated* public events and hence, such events can lead to a synchronized attack. Indeed, even an (unanticipated) increase in the likelihood of a public event alone may also trigger a crash.

Throughout we will assume that traders who are not already aware of the bubble do not observe the public event. If the public event serves to make all arbitrageurs aware of the bubble and this fact is common knowledge then the model become degenerate: we are back in a world of symmetric information in which the bubble cannot possibly survive after the public event.

6.1 Unanticipated Public Events

We consider the case where public signals arrive at a constant arrival rate λ_p. Let λ_p be sufficiently small such that the uncertainty about possible future public events alone does not trigger an immediate crash. Arbitrageurs who are aware of the bubble become more and more wary as time goes by. Therefore, they increasingly look out for signals which might cause the bubble to burst even though these signals are totally unrelated to the fundamentals. We try to capture this idea by assuming that public signals are only observed by traders who became aware of the bubble more than τ_p periods ago. Traders who are either unaware of the bubble or only recently became aware of it do not observe public signals. Alternatively, one can also envision a more general setting where unaware traders observe the public signal (sunspot) but do not attribute much importance to it. This more general viewpoint would lead us to a theoretical discussion of what constitutes the "publicness" of a public event? To avoid these interesting related theoretical puzzles, we make our assumption about τ_p. Note that the case $\tau_p = 0$ captures the special case where all aware traders observe this public event.

Even just the possibility that a public event might occur alters traders' strategies and this in turn alters the likelihood that the bubble will burst in the next instant. Instead of attacking at $t_i+\tau^{**}$, arbitrageurs already attack at $t_i+\tau^{***}$, where $\tau^{***} < \tau^{**}$. All traders who directly observe the public event can trade conditional on it. That is, they can synchronize their actions. If all traders who observe the signal attack the bubble, the bubble bursts with a strictly positive probability from the viewpoint of each individual trader. Since the cost of an instantaneous attack is zero, it is optimal for all arbitrageurs who observe the public signal to sell their assets.[16] If the selling pressure surpasses κ, the bubble bursts even though the public event carries no fundamental news. In Wall Street jargon, the public event serves as a "smokescreen" for the price correction. If the selling pressure is less than κ, the attack fails. This reveals to the attacking traders that the bubble is not ripe yet and even traders who started attacking the bubble prior to the public event invest in the market again, thereby strengthening the bubble.

Analogous to the previous sections we define the function $\psi(t|h_t)$ for any equilibrium ψ to indicate at any time t and for any history of past public events h_t, $\inf_{t_i}(t-t_i)$ where the infinimum is taken with respect to set of arbitrageurs who attack at time t. As in the previous section we focus on the equilibrium, where sustained bubbles are least likely. That is, one for which $\psi^{***}(t|h_t) \leq \psi(t|h_t)$ for all t, a given history h_t of public events and any equilibrium ψ. For $\tau_p \geq \tau^{**}$, public events can be neglected as the analysis is identical to the one in Section 5. Therefore, we focus our analysis on the case, where $\tau_p < \tau^{**}$.

Proposition 5 *In the ψ^{***}-equilibrium as defined above arbitrageur $\underline{t}_i$ always attacks at the instances of public events $t_p \geq t_i+\tau_p$. Furthermore, she attacks at all $t \geq t_i+\tau^{***}$ except in the event that the last attack failed in which case she re-enters the market for the interval $t \in (t_p, t_p+\tau^{***}-\tau_p) \cap (t_p, t_i+\tau^{1,p})$ unless a new public event occurs in the interim or after $t_i+\tau^{1,p}$. At the latter time the density that the bubble bursts for exogenous reasons or due to another public event is sufficiently high to warrant exit (even if other traders do not attack).*

After a failed attack at the latest public event t_p, even traders who started attacking prior to t_p, that is, traders with $t_i < t_p - \tau^{***}$, buy back shares. Thus, they strengthen the bubble. Nobody attacks till $t_p+\tau^{***}-\tau_p$ after a failed attack at t_p except if $t \geq t_i+\tau^{p,1}$, in which case the perceived density that the bubble bursts for exogenous

[16] Many market timers follow this strategy. For example, Richard Buch of the Seattle-based Merriman Capital Management illustratively notes: "It is like rushing out of a building when you think there's a fire. When there isn't, you go back in rather sheepishly and everyone asks 'Why are you so nervous?' But, once in a while, everyone stays in the building and there actually is a fire. The real value of market timing is getting out before a big crash." Financial Times, Weekend November 25/26th, 2000.

reasons or due to the arrival of a new public event is too high. In short, Proposition 5 illustrates how market sentiment bounces back after a failed attack.

This section highlights the point that it is much more important to focus on news events that other traders consider as possible price movers than to focus on fundamentals. As pointed out earlier, our analysis follows a more non-classical view which can be traced back to Keynes (1936).

The analysis also illustrates that increased uncertainty by itself can lead to significant price swings. This falls in line with the statement that Wall Street is more afraid of uncertainty than of the worst outcome per se.[17] An increase in the likelihood λ_p that a public event might occur in itself can trigger large price movements. Suddenly, it becomes optimal for arbitrageurs to attack the bubble even though there was no news event so far. The uncertainty about the timing of a public event can have a much bigger impact than an anticipated public event. The outcome of the 2000 US-presidential elections can be viewed in this light. Some people have argued that the *uncertainty* surrounding whether Gore or Bush won the election and when it will be resolved served as a "smokescreen" for the price correction in high-tech stocks that was anyway necessary. In our model this uncertainty about the timing allows arbitrageurs to synchronize their actions. This can lead to large price movements, while an anticipated announcement of the winner on Nov 7th, 2000 does not. The new element is that arbitrageurs can de-facto also coordinate on uncertainty about news.

Our analysis also sheds some light on the fact that there are fads and fashions in information. For example, trade figures drove the market during the 1980's. In contrast, in the late 1990's Alan Greenspan's statements moved stock prices, while trade figures were ignored.

6.2 Price Cascades and Rebounds

The most visible public events on Wall Street are probably large past price movements or a break through psychological resistance lines. In the model setup so far, we had eliminated any price impacts by assuming that any selling pressure smaller than κ does not affect the price path. We now relax this assumption in order to view price drops as public events. This enables us to illustrate how a large price decline either leads to a full blown crash or to a rebound. In the latter event the bubble is strengthened in the sense that all arbitrageurs are 'in the market' for some interval, including those who had previously exited prior to the price shock.

More formally, let a price drop by a fraction γ be viewed as a synchronization device by all arbitrageurs. We assume that this exogenous price drop occurs with a Poisson density λ'_p at the end of a random trading round t.[18] The price drop shakens

[17] Similar statements: "Wall Street hates a vacuum ..." as stated in the Economist, Friday, November 17th, 2000.

[18] This exogenous price drops are not explicitly model and can for example be due to random mood

momentum traders' mood temporarily and they are only willing to take on shares if the price is less than price after the price drop $(1-\gamma)\,e^{gt}$. If the bubble does not burst in the subsequent trading round, momentum traders regain their confidence and are willing to sell and buy at a price of e^{gt} until their absorption capacity κ is reached. Consequently, arbitrageurs who exit the market after a price drop receive \$ $(1-\gamma)\,e^{gt}$ per share. Should the synchronized attack after a price drop fail, arbitrageurs can only buy back their shares at price of e^{gt}. In other words, leaving the market even only for an instant is very costly. Hence, only traders who are sufficiently certain that the bubble will burst after the price drop will leave the stock market. More specifically, Proposition 6 shows that only traders who became aware of the mispricing more than τ'_p, will choose to leave the market and attack the bubble after a price drop. Notice, although τ'_p is derived endogenously for the subgame after a price drop, it serves the same role as τ_p in Subsection 6.1. Consequently, Proposition 6 has the same structure as Proposition 5.

Proposition 6 *There exists a unique symmetric trigger strategy equlibrium* $\left(\tau'_p, \tau^{****}\right)$. *In this equilibrium arbitrageur* $\underline{t}_i$ *exits the market after a price drop at* t'_p *if* $t'_p \geq t_i + \tau'_p$. *Furthermore, she is out of the market at all* $t \geq t_i + \tau^{****}$ *except in the event that the last attack failed in which case she re-enters the market for the interval* $t \in \left(t'_p, t'_p + \tau^{****} - \tau'_p\right) \cap \left(t'_p, t_i + \tau'^{1,p}\right)$ *unless a new public event occurs in the interim or after* $t_i + \tau'^{1,p}$. *At the latter time the density that the bubble bursts for exogenous reasons or due to another public event is sufficiently high to warrant exit (even if other traders do not attack).*

Proposition 6 shows that a price drop which is not followed by a crash leads to a sudden rebound and temporarily strengthens the bubble. In this case all arbitrageurs can rule out, that the bubble will burst within $\left(t'_p, t'_p + \tau^{****} - \tau'_p\right) \cap \left(t'_p, t_i + \tau'^{1,p}\right)$. Within this time interval, the price grows at rate of g with certainty. Consequently, all arbitrageurs re-enter the market after a failed attack and buy back shares at a price of e^{gt}, even if they have sold them an instant earlier for $(1-\gamma)\,e^{gt}$. As noted earlier the structure of Proposition and proof is the same as of Proposition 5. In the comparable case in which $\lambda'_p = \lambda_p$ and $\tau'_p = \tau_p$, $\tau^{****} < \tau^{***}$ since after a price drop the first randomly selected orders are only executed at the price of $(1-\gamma)\,e^{gt}$ instead of e^{gt} in the case of a 'regular' public event.

Note that it is important that price drops can also occur prior to time t_0. Otherwise, immediately after a price drop it is commonly known that a bubble exists and a backwards induction argument starting from $t'_p + \bar{\tau}$ would lead to an immediate collapse of the bubble.

changes by noise traders.

7 Conclusion

We confine ourselves to a few brief remarks while referring the reader to the introduction for motivation and description of our model and its main results.

We have developed a new model which serves both as a general metaphor for differences of opinion, information and belief among traders, and, more literally, as a reduced-form modeling of the temporal expression of heterogeneities amongst traders. While it is well understood that appropriate departures from common knowledge will permit bubbles to persist, we believe that our particular formulation is both neutral and parsimonious. Furthermore, since bubbles have to arise in any possible trading equilibrium for a wide range of parameter values we consider, our results suggest that departures from rational prices are persistent, and that bubbles are a robust phenomenon. The model provides a setting in which 'overreaction' and self-feeding price drops, leading to full-fledged crashes, will naturally arise. It also provides a framework which allows one to rationalize phenomena such as 'resistance lines' and fads in information gathering.

Finally we note here that many of the assumptions of our simple model may be viewed as being conducive to arbitrage. In particular, we assume that all professionals are in agreement that assets are overvalued, while arguably there are substantial differences in opinion even amongst professionals regarding the possibility that current valuations indeed reflect a new era of higher productivity growth, lower wages and inflation etc. Presumably incorporating these realistic complications would reinforce our conclusions.

A Appendix

A.1 Proof of Lemma 3

Let us denote the upper and lower bound of the support of trader $\underline{t}_j$'s beliefs about t_0 by $\left[\underline{t_0^{\text{supp}}}, \overline{t_0^{\text{supp}}}\right]$.

upper bound

$\mathbf{t_0} \leq \mathbf{t}_j = \overline{t_0^{\text{supp}}}$: Because traders become aware of the bubble after its emergence.

lower bounds

(i) $\mathbf{t_0} \geq \mathbf{t}_j - \boldsymbol{\kappa\theta} = \underline{t_0^{\text{supp}}}$ for $t_j - \kappa\theta > t - \bar{\tau}$

Suppose $\underline{t_0^{\text{supp}}} < t_j - \kappa\theta$, then with a strictly positive probability the mass of traders attacking at t is $s_{t_0,t}$ given our belief restriction that all trader $\underline{t}_{i'} \leq \underline{t}_j$ also attack at t. This corresponds to the aggregate selling pressure $s_{t_0,t} > \kappa$. In this case, traders would experience a (strict) price decline at t and thus trader $\underline{t}_j$ has an incentive to attack an instant earlier. Thus $\underline{t_0^{\text{supp}}} \geq t_j - \kappa\theta$ if $\psi(t) > 0$.

In the special case $\psi(t) = 0$, trader t_j cannot attack earlier than t_j and thus an al-

ternative argument has to be employed. In this case the bubble can only still exist prior to t_j and $s_{t_0,t_j} > \kappa$ if at t' sufficiently close to t_j, not all 'aware' arbitrageurs were attacking the bubble, i.e. $\psi(t) > 0$.[19] However, this is not optimal for them since all of them have an incentive to preempt the possible crash at t_j.
Suppose $\underline{t_0^{\text{supp}}} > t_j - \kappa\theta$, then the mass of traders attacking at t and thus the corresponding selling pressure is $s_{t_0,t} < \kappa$ with probability one. Hence, it pays for trader $\underline{t}_j$ to delay her attack. This is the case, since for trader $\underline{t}_j$ it is not possible that $s_{t_0,t_j+\tau_j}$ will jump to κ in the next instant. For trader $\underline{t}_j$ the maximum increase in $s_{t_0,t+dt}$ is at most dt. Consequently, $t_j - \kappa\theta = \underline{t_0^{\text{supp}}}$.
(ii) $\mathbf{t_0} \geq \mathbf{t} - \bar{\boldsymbol{\tau}} = \underline{t_0^{\text{supp}}}$ for $t_i - \kappa\theta < t - \bar{\tau}$
Since the bubble bursts at $t_0 + \bar{\tau}$, each trader can immediately infer from the existence of the bubble at time t that $t_0 \geq t - \bar{\tau}$.

A.2 Proof of Proposition 1

for $\mathbf{t} - \mathbf{t}_j \leq \bar{\boldsymbol{\tau}} - \boldsymbol{\theta\kappa}$
Since trader $\underline{t}_j$ knows the equilibrium $\psi(t)$ function, she is aware that her time lag between awareness and attacking the bubble is the shortest. Given the above Lemma, when agent $\underline{t}_j$ considers attacking the first time, she thinks that t_0 is distributed over $[t_j - \theta\kappa, t_j]$. Given that the prior is exponentially distributed with $F(t_0) = 1 - e^{-\lambda t_0}$, the prior probability that $t_0 \in [t_j - \theta\kappa, t_j]$ is $e^{-\lambda(t_j - \theta\kappa)} - e^{-\lambda t_j}$. Since arbitrageurs become aware of the mispricing in a uniform manner the conditional density of t_0 is therefore given by $\frac{\lambda e^{-\lambda t_0}}{e^{-\lambda\left(t_j - \theta\kappa\right)}\left(1 - e^{-\lambda\theta\kappa}\right)}$. For $t_0 = t_j - \theta\kappa$ the density is $\frac{\lambda}{1 - e^{-\lambda\theta\kappa}}$. For trader $\underline{t}_j$, the density that the bubble bursts in the next instant is therefore always smaller or equal to $\frac{\lambda}{1 - e^{-\lambda\theta\kappa}}$. By Lemma 1, it is never optimal to attack at this point if $\frac{\lambda}{1 - e^{-\lambda\theta\kappa}} > g - r + c$. Since $\theta\kappa > \frac{-\ln\left(1 - \frac{\lambda}{g - r + c}\right)}{\lambda}$, this necessary equilibrium condition is violated.
for $\mathbf{t} - \mathbf{t}_i > \bar{\boldsymbol{\tau}} - \boldsymbol{\theta\kappa}$
As $\tau_i = t - t_i$ approaches $\bar{\tau}$, trader $\underline{t}_i$'s perceived density that the bubble bursts in the next instant is at least $\frac{\lambda}{1 - e^{-\lambda\left(t_j - t + \bar{\tau}\right)}} = \frac{\lambda}{1 - e^{-\lambda(\bar{\tau} - \tau_i)}}$, which is increasing in τ_i. By Lemma 1, trader $\underline{t}_i$ attacks never attacks before τ_i where $\frac{\lambda}{1 - e^{-\lambda(\bar{\tau} - \tau_i)}} \geq g - r + c$, that is

Since the mass of traders attacking at $t_0 + \bar{\tau}$ is smaller than $\frac{1}{\theta}\frac{-\ln\left(1 - \frac{g - r + c}{\lambda}\right)}{\lambda}$ and hence smaller than κ the bubble does not burst for endogenous reasons.

[19] This argument relies on the seniority condition implied by the definition of a "trading equilibrium." It assumes that each attacking trader beliefs that all traders who became aware of the bubble before her also attack the bubble.

A.3 Proof of Proposition 3

Proof. Let us conjecture that all rational arbitrageurs leave the market at $t_i + \tau^{**}$. Given this symmetric trigger strategy profile, the bubble will burst at $t = t_0 + \tau^{**} + \theta\kappa$. Since this is known by all arbitrageurs, they also know that when the bubble bursts its size is $e^{gt}\left[1 - (1-\beta)\, e^{-(g-r)(\tau^{**}+\theta\kappa)}\right]$. Consequently, we can focus on the attack condition

$$\frac{\lambda}{1 - e^{-\lambda(t_i+\tau^{**}+\theta\kappa - t)}} \geq \frac{(g-r)+c}{1-(1-\beta)\, e^{-(g-r)(\tau^{**}+\theta\kappa)}}$$

Rearranging attack for

$$t \geq t_i + \tau^{**} + \theta\kappa - \frac{-\ln \frac{(g-r)-\lambda+\lambda(1-\beta)e^{-(g-r)(\tau^{**}+\theta\kappa)}}{(g-r)+c}}{\lambda}$$

provided that $\tau^{**} \geq 0$. Solving for the critical $\tau^{**} = \tau^{**} + \theta\kappa - \frac{-\ln \frac{(g-r)-\lambda+\lambda(1-\beta)e^{-(g-r)(\tau^{**}+\theta\kappa)}}{(g-r)}}{\lambda}$, yields

$$\tau^{**} = \frac{-\ln\left\{\frac{(g-r+c)e^{-\lambda\theta\kappa}-(g-r+c)+\lambda}{\lambda(1-\beta)}\right\}}{(g-r)} - \theta\kappa$$

Hence the bubble bursts at $t_0 - \frac{1}{g-r}\ln\left\{\frac{\lambda-(g-r+c)\left[1-e^{-\lambda\theta\kappa}\right]}{\lambda(1-\beta)}\right\}$ if $\tau^{**} \geq 0$ and at $t_0 + \theta\kappa$ otherwise. That is, the bubble bursts at

$$t_0 + \max\left\{\theta\kappa, \frac{-\ln\left\{\frac{\lambda-(g-r+c)\left[1-e^{-\lambda\theta\kappa}\right]}{\lambda(1-\beta)}\right\}}{(g-r)}\right\}.$$

The bursting time is well defined, since the initial condition $\theta\kappa < \left(\frac{1}{\lambda}\right)\left[-\ln\left(1 - \frac{\lambda-\lambda(1-\beta)e^{-(g-r)\bar{\tau}}}{g-r+c}\right)\right]$ guarantees that $\frac{\lambda-(g-r+c)\left[1-e^{-\lambda\theta\kappa}\right]}{\lambda(1-\beta)} \geq 0$. Notice that for $\beta = 0$, $-\ln\left[\frac{\lambda-(g-r+c)\left[1-e^{-\lambda\theta\kappa}\right]}{\lambda(1-\beta)}\right]\left\{\frac{1}{g-r}\right\} \geq \theta\kappa$ for any $\theta\kappa$ since (a) for $\theta\kappa = 0$, $t_0 + \theta\kappa = t_0 - \ln\left[\frac{\lambda-(g-r+c)\left[1-e^{-\lambda\theta\kappa}\right]}{\lambda(1-\beta)}\right]\left\{\frac{1}{g-r}\right\}$ and (b) $\frac{\partial\left\{-\ln\left[\frac{\lambda-(g-r+c)\left[1-e^{-\lambda\theta\kappa}\right]}{\lambda(1-\beta)}\right]\left\{\frac{1}{g-r}\right\}\right\}}{\partial(\theta\kappa)} > 1$ for all relevant $\theta\kappa$.

For $\tau_i > (<)\ \tau^{**}$ any arbitrageur $\underline{t}_i$ has a strict incentive (not) to attack. By inspection of the attack condition (*) $\pi^c\left(t|\underline{t}_i\right) > \frac{(g-r)+c}{1-(1-\beta)E\left[e^{-(g-r)(t-t_0)}|\underline{t}_i,t\right]}$, one can see that for $t = t_0 + \theta\kappa + \tau^{**}$, all arbitrageurs who became aware of the mispricing before (after) $\underline{t}_i$, it is optimal to be out of (in) the market. ■

Lemma 4 *There does not exist an equilibrium in which an agent attacks prior to* $t_i + \tau^{**}$*, that is, for any equilibrium* $\tilde{\psi}$, $\min_t \tilde{\psi}(t) < \tau^{**}$.

Proof. Suppose there exists an equilibrium $\tilde{\psi}$ with $\min_t \tilde{\psi}(t) < \tau^{**}$. Consider the trader at which function $\tilde{\psi}(t)$ achieves its minimum over t and call her $\underline{t}_j$. The argument can be extended to the case where $\min_t \tilde{\psi}(t)$ does not exist. Let us denote the $\arg\min_t \tilde{\psi}(t)$ by $\tilde{\underline{t}}$ and $\tilde{\underline{\tau}}_j$ the time lag after which trader $\underline{t}_j$ attacks the first time.
Step 1: There exists a sequence $t^n \to \tilde{\underline{t}} = \tilde{t}_j + \tilde{\tau}_j$ s.t. $\tilde{\psi}(t) \to \tilde{\psi}(\tilde{\underline{t}})$.
Lemma 3 shows that the support of arbitrageur $\tilde{\underline{t}}_j$'s posterior of t_0 has to be $\left[\tilde{t}_j - \theta\kappa, \tilde{t}_j\right]$, i.e. $\Pr_{\tilde{j}}\left(t_0 \in \left[\tilde{t}_j - \theta\kappa, \tilde{t}_j\right] | \tilde{F}_t, \tilde{t}_j, t\right) = 1$ in any equilibrium. Recall $\tilde{F}_t$ denotes the event where the bubble does not burst prior to t. Suppose that $\liminf_{\tilde{\psi}(t) \to \tilde{\psi}(t_j)} = \tilde{\tau}_{j,t} + \eta$, then trader $\tilde{\underline{t}}_j$'s support for the distribution of t_0 would be $\left[\tilde{t}_j - \theta\kappa - f(\eta), t_j\right]$, with $f(\eta) > 0$. That is, he could not infer from the existence of bubbles that $t_0 \geq \tilde{t}_j - \theta\kappa$, which contradicts Lemma 3.
Step 2: Recall that $\psi(t)$ refers to the attacking trader who is the last to become aware of the bubble. Let us denote her ex-ante payoff from attacking at $s > t$ by $V_t(s|\psi)$ and the marginal payoff change from delaying to attack by,

$$v(t|\psi) = \left.\frac{dV_t(s|\psi)}{ds}\right|_{s=t}$$

Instead of focusing on the interim expectations $E_{t_0}[V_t | t_i, F_t]$ conditional on the existence of the bubble, it is more convenient to focus on the ex-ante expectations V_t. This can be done without loss of generality since for the event F_t^c, that is, for the case that the bubble has already burst, the decision is irrelevant because the payoff is a constant zero.
$\psi^{**}(t) = \tau^{**}$ for all t, corresponds to the equilibrium where all traders attack after τ^{**}.
$\tilde{\psi}(t)$ corresponds to potential alternative equilibrium which is described by the attack strategies $\tilde{\sigma}$.
At $\underline{t} := \arg\min_t \tilde{\psi}(t)$, trader $\tilde{\underline{t}}_j$ attacks after a time lag of $\underline{\tau} := \tau_j = \min_t \tilde{\psi}(t)$ periods. If every trader were to attack after $\underline{\tau}$, then the function $\underline{\psi}(t) = \underline{\tau}$ for all t, describes the corresponding outcome.
Since $\psi^{**}(t)$ describes an equilibrium (and assuming differentiability), the marginal payoff of waiting conditional on the event F_t that the bubble still exists, is zero. That is $v(t|\psi^{**}) \Pr_{\psi^{**}(t)}(F_t) = 0$. Since $\Pr_{\psi^{**}(t)}(F_t) > 0$, the ex-ante marginal payoff is also $v(t|\psi^{**}) = 0$.
Suppose each arbitrageur were to attack $\underline{\tau}$ periods after she becomes aware of the bubble. In this case the endogenous selling pressure at t in the $\underline{\psi}$-strategy profile would coincide with the endogenous selling pressure in the ψ^{**}-equilibrium at $t + \tau^{**} - \underline{\tau}$. However at $t + \tau^{**} - \underline{\tau}$ each arbitrageurs estimate of the size of the bubble is strictly

larger than the estimates at t. Thus $v\left(t|\underline{\psi}\right)\Pr_{\underline{\psi}(t)}(F_t) > v\left(t|\psi^{**}\right)\Pr_{\psi^{**}(t)}(F_t) = 0$. That is $v\left(t|\underline{\psi}\right) > 0$.

Step 1 of the proof showed that the candidate equilibrium described by $\tilde{\psi}(t)$ has the property that there exists a sequence $t^n \nearrow \underline{t}$, $\tilde{\psi}(t^n) \searrow \tilde{\psi}(\underline{t}) = \underline{\psi}(\underline{t}) = \underline{\tau}$.

For any sequence $t^n \nearrow \underline{t}$ and $\tilde{\psi}(t^n) \searrow \tilde{\psi}(\underline{t})$, $v\left(t^n|\tilde{\psi}\right) \to v\left(\underline{t}|\tilde{\psi}\right)$.

Hence, there exists a $\varepsilon > 0$, s.t. for all $t \in (\underline{t} - \varepsilon, \underline{t})$, $v\left(t|\tilde{\psi}\right) > 0$. In words, in the potential equilibrium $\tilde{\sigma}$, trader $\underline{t}_j$ has a strict incentive to delay attacking. Consequently, there does not exist any $\tilde{\sigma}$ equilibrium, with $\underline{\tau} < \tau^{**}$. ■

A.4 Proof of Proposition 5

Proof.

Attack at public events

From each arbitrageur's point of view, a bubble bursts with strictly positive probability at each t_p if others attack at t_p, too. Given this belief and the fact that an instantaneous attack is costless, it is always an equilibrium that each trader who observes the public event attacks the bubble at t_p.

Prior to arrival of public events

Prior to $t_0+\theta\kappa+\tau_p$ the public event is only observed by less than κ traders and thus it has no impact on the bubble. From $t_0+\theta\kappa+\tau_p$ onwards each public event bursts the bubble. That is, the bubble can also burst due to public events with an additional Poisson density of λ_p from $t_0+\theta\kappa+\tau_p$ onwards. No arbitrageur $\underline{t}_i$ knows t_0 and the perceived density that public event occurs which bursts the bubble is $\lambda_p F\left(t \geq t_0+\theta\kappa+\tau_p|\underline{t}_i, t\right)$. Recall that $F\left(t \geq t_0+\theta\kappa+\tau_p|\underline{t}_i, t\right)$ is the probability that $t \geq t_0+\theta\kappa+\tau_p$. The attack condition generalizes to

$$\begin{aligned}
&\frac{\lambda}{1-e^{-\lambda(t_i+\tau^{***}+\theta\kappa-t)}}\left[1-(1-\beta)\,e^{-(g-r)(\tau^{***}+\theta\kappa)}\right]+ \\
&+\lambda_p F\left(t \geq t_0+\theta\kappa+\tau_p|\underline{t}_i, t\right)\left[1-(1-\beta)\,E\left[e^{-(g-r)(t-t_0)}|\underline{t}_i, t\right]\right]\left(1-\alpha_{p,t,t_0}\right) \\
\geq\ &(g-r)+c.
\end{aligned}$$

If the bubble bursts at $\tau^{**}+\theta\kappa$ only, then its size is $e^{gt}\left[1-(1-\beta)\,e^{-(g-r)(\tau^{***}+\theta\kappa)}\right]$ which is reflected in the first term. If on the other hand the bubble bursts prior to this date, due to the arrival of a public event trader $\underline{t}_i$'s expected size is $e^{gt}[1-(1-\beta)\,E[e^{-(g-r)(t-t_0)}|\underline{t}_i, t]]$. The advantage of attacking prior to the public event has to be multipied by $(1-\alpha_{p,t})$, since a trader can still leave the market at the pre-crash price with probability $\alpha_{p,t}$ even after the public event. Notice that $\alpha_{p,t}$ is the (expected) fraction of orders executed at the pre-crash price $\kappa - E\left[\frac{1}{\theta}\left(t-t_0-\tau^{***}\right)|t_i, t\right]$ over all orders submitted immediately after a public event $\frac{1}{\theta}\left(\tau^{***}-\tau_p\right)$.

Rather than attempting to solve for τ^{***} in closed form, we show that τ^{***} exists and $\tau^{***} < \tau^{**}$. Consider the following procedure of iterative removal of trigger

strategies. Suppose attacking from $t_i + \tau^1$ onwards is a best response to the conjecture that other agents never attack. Note that $\tau^1 < \infty$ since the bubble bursts latest at $t_0 + \bar{\tau}$ for exogenous reasons. Suppose attacking from $t_i + \tau^n$ is a trigger best response to the conjecture that all other agents attack from $t_i + \tau^{n-1}$ onwards. By restricting arbitrageurs beliefs about others' strategies to *symmetric* trigger strategies, the strategies become strategic complements. Therefore the sequence (τ^n) is weakly decreasing. Let $\tau^{***} := \lim_{n\to\infty} \tau^n$. Since agents' payoff are continuous in the times at which they attack, it follows that the strategy profile in which all agents attack at τ^{***} yields a perfect Bayesian equilibrium. More formally, let the payoff of attacking from $t_i + \tau_i$ onwards given that all others attack after τ_{-i} be denoted by $V(\tau_i|\tau_{-i})$. By definition $V(\tau^n|\tau^{n-1}) - V(\tau|\tau^{n-1}) \geq 0$ for all τ. By continuity of $V(\cdot)$, for $n \to \infty$, $\lim_{n\to\infty} V(\tau^{***}|\tau^{***}) - V(\tau|\tau^{***}) \geq 0$.
It is easy to see that $\tau^{***} < \tau^{**}$. For the special case where there are no public events, that is when $\lambda_p = 0$, the attack condition is identical to the one used in Proposition 3 and hence $\tau^{***} = \tau^{**}$. For $\lambda_p > 0$, the incentive to exit the market is strictly higher for each arbitrageur. Consequently, $\tau^{***} < \tau^{**}$.
Strategy after failed attack
A failed attack at t_p reveals that $t_0 > t_p - \tau_p - \theta\kappa$, since less than κ arbitrageurs attacked. Given the strategy to attack with a delay of τ^{***} periods, fewer arbitrageurs would attack prior to $t = t_p + \tau^{***} - \tau_p$ than at t_p. Consequently, the bubble will not burst till $t_p + \tau^{***} - \tau_p$. All arbitrageurs, who observed the failed attack at t_p know this and thus re-enter the market and ride the bubble till $t_p + \tau^{***} - \tau_p$, modulo the qualifications below. After $t = t_p + \tau^{***} - \tau_p$, the analysis coincides with a setting without a public event at t_p. It should be noted that there are two exceptional cases where arbitrageurs leave the market again prior to $t_p + \tau^{***} - \tau_p$. First, they attack again should a new public event occur and secondly after $t = t_i + \tau^{p,1}$, when the density that the bubble will burst for exogenous reasons or due to another random public event is so high, that the arbitrageur wishes to exit the market. ■

A.5 Proof of Proposition 6

Lemma 5 *There exists a τ'_p such that all traders $\underline{t}_i$ who became aware of the mispricing prior to $t'_p - \tau'_p$ leave the market and all other arbitrageurs stay in the stock market.*

Proof. Let us denote τ^{****} the time elapsed after which trader leave the market in the absence of price drops attack τ^{****}. Exiting the market after a price drop is optimal if and only if

$$\Pi'\left(t'_p|t_i\right)\left[(1-\gamma)e^{gt} - E\left[v_t|t,t_i\right]\right]\alpha'_{p,t} + \left(1 - \Pi'\left(t'_p|t_i\right)\right)\left[(1-\gamma)e^{gt} - e^{gt}\right] \geq 0.$$

$\Pi'\left(t'_p|t_i\right)$ denotes the trader $\underline{t}_i$'s probability that the bubble will bursts after a price drop at t'_p. If the bubble does not burst, then all arbitrageurs sell their shares at the price of $(1-\gamma)e^{gt}$. If the bubble does burst, $\left[\kappa - \frac{1}{\theta}\left(t'_p - t_0 - \tau^{****}\right)\right]$ orders are executed at $(1-\gamma)e^{gt}$ while the remaining orders are executed at a price equal to v_t. The term $\alpha'_{p,t} = \frac{\theta\kappa - \left(t'_p - t_0 - \tau^{****}\right)}{\tau'_p - \tau^{****}}$ reflects the fact that only the first randomly selected order are executed at $(1-\gamma)e^{gt}$. Rearranging the attack condition yields

$$\Pi'\left(t'_p|t_i\right) \geq \frac{\gamma}{\alpha'_{p,t} + \gamma\left(1-\alpha'_{p,t}\right) - \alpha'_{p,t}\frac{E[v_t|t,t_i]}{e^{gt}}}.$$

Given the conjecture that all $\underline{t}_i$-arbitrageurs with $t_i \geq t'_p - \tau'_p$ are out of the market, $\Pi'\left(t'_p|t_i\right) = F\left(t_0 \leq t - \theta\kappa - \tau'_p|t, t_i\right)$ which is decreasing in t_i. Since futhermore $E[v_t|t, t_i]$ is increasing in t_i, there exists a critical level τ'_p such that for all arbitraguers with $t_i \leq t'_p - \tau'_p$ the attack condition is sastified and for all $t_i > t'_p - \tau'_p$ not, thereby validating the conjecture. ■

Proof of Proposition 6.
After establishing the critical value τ'_p in Lemma XX, the remaining proof of Proposition 6 is analogous to the proof of Proposition 5. The only difference is that after the price drop, the first orders are only executed at a price of $(1-\gamma)e^{gt}$ instead of the pre-crash price e^{gt}. ■

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