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Defining residual risk-sharing opportunities: Pooling world income components*

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Summary

We construct a new method of decomposing the variance of national incomes into components in such a way as to indicate the most important 'residual' risk-sharing opportunities among peoples of the world. The risk-sharing opportunities we study are nonsystematic risk-sharing opportunities. These are the risk-sharing opportunities that would remain if systematic risk were already shared, see Athanasoulis and Shiller (2000). The new method developed here uses a simpler approach to deriving the components based on pure variance reduction. With the new method, the income component securities are derived in terms of eigenvectors of a transformed variance matrix of world incomes, but with this method, the transformation is to use the residuals when incomes are regressed on world income instead of deviations of incomes from average world income as in Athanasoulis and Shiller (2001). The method is applied using Summers-Heston (1991) data on national incomes for large countries 1950–1990, using two different methods of estimating variances.

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1. Introduction

In this paper, we develop and extend a method (Athanasoulis and Shiller 2000, 2001) for characterizing the risk structure of world incomes and for producing definitions of a small number of securities (contracts) that will allow us to create new markets for much of this risk. The securities we advocated in these earlier papers, called Claims on Linear Income Combinations (CLICs), would allow very important advances in our risk management. In this paper, in contrast to the earlier papers, we study the use of CLICs to manage nonsystematic risk, that is, risk unrelated to world risk, risk that has a zero price in the market. Our extended method allows us to achieve the important objective of specifying contracts for a world in which world risk is already traded.

An important motivation for deriving these components is the simplicity these contracts afford to individuals for risk sharing. Whenever new contracts are constructed, the success of the contracts depends critically on how easily one can understand the contracts for their use. Thus, verifying whether contracts that have a high probability for success, from the standpoint of simplicity, offer enough risk-sharing opportunities to guarantee their success, is very important.

It follows from the analysis in Athanasoulis and Shiller (2000) that if a market for systematic risk (the world portfolio) exists, then the contracts constructed below are the remaining optimal CLICs. It also follows from Athanasoulis and Shiller (2000) that this contract design method is robust to errors in assumptions about preference parameters the contract designer may make.

Our method is related to principal components analysis applied to the residuals of national incomes (strictly speaking, gross domestic products, GDPs), when regressed on World Income. Our data consist of the Penn World Table data on annual real *per capita* GDPs for the twelve largest (in terms of 1990 GDP) countries 1950–1990, measured in 1985 U.S. dollars, see Summers and Heston (1991).

A product of our analysis is a set of indexes (that is, linear combinations) of national incomes, designed to be used as the basis of settlement for CLICs. We will refer to the kind of CLICs derived in this paper as pooling world income components (pooling-WICs) securities. The kind of CLICs in our earlier paper, (2001), may now be called premium world income components (premium-WICs) securities to distinguish them. The premium-WICs also had a price that initially was zero, but there was also a positive insurance premium that the longs paid the shorts.

We would expect the pooling-WIC contracts to be traded on securities markets just as other securities are traded today. They might also be called pooling-WIC futures contracts and be traded at futures exchanges. The securities are not best thought of as insurance policies, since the countries are pooling risks not paying a premium for others to assume risk. Individuals in countries will trade risk in order to reduce risk and none of it will be to shift risk from some countries to others for a premium.

The theoretical model constructed in this paper is a one-period model. Individuals, (countries), calculate their present value of national income over the horizon they wish to share risk for, then buy and sell the contracts to minimize their variance. Though in general, as new information becomes available, the individuals may wish to re-trade the contracts, we do not provide a theory for that here.† We instead wish to provide a simple model with some simple empirical calculations to see what the best risk-sharing securities will be and what the benefit from these securities are.

The theoretical model draws on some of the security design literature. In particular, there are similarities to the methods developed by Athanasoulis and Shiller (2000, 2001), Cuny (1993), Duffie and Jackson (1989) and Demange and Laroque (1995). All of the above methods on contract design are utility based methods in which the authors assume some preferences for the individuals and then find the most welfare improving contract by some metric. The method in this paper is utility free and thus is free from any parametric assumptions about utility functions. The contracts allow agents to share in nonsystematic risk, which would be the only useful contracts if a market for world income were already in existence. For such arguments, see Athanasoulis and Shiller (2000). One can also find arguments for simplicity and robustness of contract definition for these contracts in Athanasoulis and Shiller (2000).

Our empirical section also draws on a vast literature on international risk sharing. Many authors have calculated the welfare gains from risk sharing across countries in the world.‡ Our measure of benefit will be preference free and is very similar to Athanasoulis and van Wincoop (2001). In general, researchers assume some preferences and stochastic process for consumption (or income), and calculate the certainty equivalent welfare gain from risk sharing. This type of exercise depends crucially on

† For a theory with the possibility of re-trading of contracts see Athanasoulis and Shiller (2001).

‡ See for example Athanasoulis (1995), Athanasoulis and Shiller (2001), Athanasoulis and van Wincoop (2000, 2001), Cole and Obstfeld (1991), Davis and Willen (2000), Lewis (2000), Tesar (1995) and van Wincoop (1994, 1999).

the assumed utility function and the process income follows. The measure we have in this paper is preference free as if individuals are risk averse, they will always wish to diversify away nonsystematic risk. As such the measures of risk reduction can give an unbiased view of the importance of these markets.

Our study of residual risk-sharing opportunities among national incomes is potentially very important, since national incomes are measures of total economic welfare of the countries, and since there have been historically large variations in real national incomes. In Shiller (1993a) (see also Shiller (1993b)) it was proposed that markets be established for long-term, even perpetual, claims on national incomes; it was argued that, despite some potential problems, such markets are indeed feasible. Here, our pooling-WIC securities will be defined as finite-term, T-year, claims on the indexes (linear combinations) of national incomes defined here. In our empirical work below we will consider securities with T of both ten and forty years. Ideally, there would eventually be securities for an array of horizons and for perpetual securities, so that people with different circumstances in terms of years of life expected or number of heirs could find a security tailored to their interests.

Our approach to defining income indices, so that long-term claims on the indices can be traded for risk management is a pure variance reduction strategy. With this strategy, we assume that individuals in each country are interested only in reducing the variance of their income, and we constrain the *ex ante* price of the securities to equal zero. We seek to define contracts such that excess demand is zero and the securities have a zero price, for countries that seek only to reduce risk in trading these securities. We then seek to define a small number of securities that allow for the most overall risk reduction subject to the restriction on the number of securities. With this strategy, the method of defining securities has a clear and simple relation to principal components analysis: it turns out that the optimal securities are defined in terms of eigenvectors of a sort of variance matrix of residuals produced when national incomes are regressed on world income, rather than in terms of the variance matrix of deviations of incomes from world average income, as in Athanasoulis and Shiller (2001).

In section 2 below, we discuss how to apply our method of defining the CLIC securities to the data. Two methods of estimating variance matrices of national incomes are also used, a method that uses sample moments directly and a method that uses strong prior restrictions to estimate. In section 3 we present results for both ten countries (unrestricted variance matrix) and twelve countries (restricted variance matrix). In section 4 we interpret these results as suggesting genuine opportunities for important new markets and section 5 concludes.

2. Definition of contracts and risk structure

In each of the new markets to be created, CLICs are to be traded that represent claims on a stream of index values according to a standard contract specified by the securities or futures exchange. We assume that there are N markets created indexed by $n = 1, \ldots, N$. At the beginning of a contract in the nth market, at contract year 0, the long in the contract is paid, each contract year from $t = 1, \ldots, T$, a random amount D_{nt} from the short in the contract, the year t "dividend" paid on the CLIC security, to be determined in year t according to a linear formula defined in the contract in year 0. The dividend D_{nt} is our nth CLIC security at year t, a linear function of national incomes accruing to year 0 populations in that year. Note that this dividend can be either positive or negative. National incomes in year t accruing to year 0 populations (which we will refer to here loosely as national incomes) are taken to be *per capita* gross domestic products in year t times the corresponding populations at year 0. We will assume that each contract signer individually can be expected to earn his or her share of the *per capita* national income in subsequent years from sources other than the contracts we define here. The linear function of national incomes specifying the dividend is defined in the initial contract year 0 so that the present value over T years of the function is defined to have an expectation, conditional on information at year 0, of 0. We are assuming here that public expectations of future real *per capita* national incomes are objective public knowledge, so that contracts can be written in terms of these expectations, though in practice some rough proxy for the expectations would have to be used by contract designers. We study the contracts from the standpoint of the year they are initiated, year 0, only.

Let us define the $1 \times J$ random vector $\mathbf{X}$ whose jth element, $j = 1, \ldots, J$, is the present value in year 0 (the year the contract is made) of real *per capita* national income for country j for the years 1 through T minus the expectation at year 0 of this present value, all times population of country j in year 0.† Thus, taking E_0 as the expectations operator conditional on information available at year 0, we have that $E_0\mathbf{X} = \mathbf{0}$ and the conditional variance matrix for T-year present value of national incomes accruing to current populations is $\Sigma = E_0(\mathbf{X}'\mathbf{X})$.

The nth CLIC security has a present value of dividend payout D_n where $D_n = \mathbf{X}\mathbf{A}_n$ and $\mathbf{A}_n$ is a $J \times 1$ element vector whose jth element is the fraction of national income of country j that is

† In practice, we use real gross domestic product to proxy for national income. We use a constant real discount rate, the same for all countries, equal in our empirical work below to 2%.

paid as the dividend on one security in market n. Note that the dividend can be either positive or negative. Assuming that there will be N different kinds of securities traded, $0 < N < J$, let us create an N-element vector $\mathbf{D}$ whose nth element is D_n and a $J \times N$ matrix $\mathbf{A}$ whose nth column is $\mathbf{A}_n$. Then, $\mathbf{D} = \mathbf{XA}$, and $\mathbf{D}$ will be the present value of our desired vector of the index values.

Let us suppose that the contract weights $\mathbf{A}$ defining the securities are normalized so that $E(\mathbf{D}'\mathbf{D}) = \mathbf{I}$, where $\mathbf{I}$ is the $N \times N$ identity matrix. This normalization means that the variance of the present value of T years of dividends (dividends measured in thousands of 1985 dollars), summing from $t = 1, \ldots, T$, is one, and the covariances of the present value of T years of dividends with the present values of T-years of dividends of all other markets are zero. This normalization has no effect on the securities' ability to hedge risk. The normalization will have the effect of tending to make the elements of $\mathbf{A}$ very small, so that contract size is suitable for trading by individuals.

3. Index design method: pure variance reduction

With this method we seek to design income component securities, (pooling-WICs), whose price defined at the date the contract begins, $t = 0$, is zero. Designing contracts whose price is zero initially is analogous to underwriters' designing bonds to sell at par on issue. Note that since we have demeaned national incomes, trading in the zero-price contracts at the initial date has no effect on one's expected, as of that date, present value of future income.

A representative individual in country j, seeking at year 0 to hedge his or her income risk, can minimize the variance of T-year present value of income in terms of the N securities by regressing minus his or her share in the T-year present value of national income of country j onto the N T-year present values of dividends. The regression is as follows:

$$-y_j = \alpha + q_j^{ind}\mathbf{D} + \varepsilon_j \tag{1}$$

where a superscript *ind* stands for individual and y_j is the *per capita* present value of income of an individual in country j. The vector of the sum across all individuals in country j of theoretical regression coefficients is $\mathbf{q}_j = -\mathrm{E}_0(\mathbf{D}'\mathbf{D})^{-1}\mathrm{E}_0(\mathbf{D}'\mathbf{X}_j)$. Since $\mathrm{E}_0(\mathbf{D}'\mathbf{D}) = \mathbf{I}$, $\mathbf{q}_j = -\mathrm{E}_0(\mathbf{D}'\mathbf{X}_j) = -\mathbf{A}'\Sigma_j$ where Σ_j is the jth column of Σ. The optimal hedge for country j (individuals in country j considered together) will be to purchase a number $\mathbf{q}_{nj}$ of the nth security so that the unexpected component of that country's income is offset as well as possible by opposite dividends in the

CLIC, minimizing the variance of the combined incomes.† Let us combine the J vectors $\mathbf{q}_j, j = 1, \ldots, J$ into a $N \times J$ matrix $\mathbf{q}$ whose jth column is $\mathbf{q}_j$, and so $\mathbf{q} = -\mathbf{A}'\Sigma$.

Let us now infer how designers of new markets might construct the N securities in such a way that they would allow the best possible compromise over the J countries, for the purpose of allowing them to hedge well. Obviously, any given country would prefer that a market be set up specifically for hedging risks to that country's income, but such a market might not serve other countries well. To achieve a compromise, we want to minimize a weighted average of the various countries' hedging error. This means that the designer must select the matrix $\mathbf{A}$ (select terms of the contract) to minimize, by some metric, the combined errors made by everyone. The metric for the combined expected squared errors that we will use is:

$$S = tr(\mathbf{w}\mathrm{E}_0((\mathbf{X}+\mathbf{Dq})'(\mathbf{X}+\mathbf{Dq}))) \tag{2}$$

where $\mathbf{w}$ is a diagonal matrix with strictly positive elements along the diagonal. S is the expected squared errors for each country j weighted by w_j (the jth diagonal element of $\mathbf{w}$) and summed across countries. In our empirical work, we will make $\mathbf{w} = \mathbf{I}$ so that all countries have the same weight.

Now, note that $S = tr(\mathbf{w}\mathrm{E}_0((\mathbf{X}+\mathbf{Dq})'(\mathbf{X}+\mathbf{Dq}))) = tr(\mathbf{w}\Sigma) - tr(\mathbf{A}'\Sigma\mathbf{w}\Sigma\mathbf{A})$. To minimize S, we must maximize $tr(\mathbf{A}'\Sigma\mathbf{w}\Sigma\mathbf{A})$ subject to the constraint $\mathbf{A}'\Sigma\mathbf{A} = \mathbf{I}$. Moreover, we have an additional constraint that the total positions are zero; for every short there must be a long and markets clear; this constraint represents the essential motivation in our analysis that we are looking for risk-sharing opportunities, not just ordinary principal components of income. Thus, we have that:

$$\mathbf{q}\iota = \mathbf{0} \tag{3}$$

where ι is a $J \times 1$ vector of ones.

Let us first solve this maximization problem for the case of only one market, where the matrix $\mathbf{A}$ is a column vector. To maximize subject to the two constraints $\mathbf{A}'\Sigma\mathbf{A} = 1$ and $\mathbf{A}'\Sigma\iota = 0$ we set up the Lagrangian L:

$$L = \mathbf{A}'\Sigma\mathbf{w}\Sigma\mathbf{A} - (\mathbf{A}'\Sigma\mathbf{A} - 1)\lambda - (\mathbf{A}'\Sigma\iota)\mu \tag{4}$$

where λ and μ are Lagrange multipliers for the two constraints. Differentiating with respect to $\mathbf{A}$, we derive the first order

† $\mathbf{q}_{nj}$ is measured in units of number of contracts for each country, so that $\mathbf{q}_{nj}$ will presumably be a very large number, in contrast to the very small value of $\mathbf{A}_{nj}$.

condition:

$$\frac{\partial L}{\partial \mathbf{A}} = 2\Sigma\mathbf{w}\Sigma\mathbf{A} - 2\Sigma\mathbf{A}\lambda - \Sigma\iota\mu = 0. \quad (5)$$

Premultiplying the above equation by $\mathbf{A}'$, and using the facts that $\mathbf{A}'\Sigma\iota = 0$ and $\mathbf{A}'\Sigma\mathbf{A} = 1$, we find that $\lambda = \mathbf{A}'\Sigma\mathbf{w}\Sigma\mathbf{A} = \mathbf{qwq}'$; this is the total weighted variance reduction, the weighted sum of the variance reductions across all countries. Premultiplying (5) by ι' and using again that $\mathbf{A}'\Sigma\iota = 0$, we find that $\mu = 2\iota'\Sigma\mathbf{w}\Sigma\mathbf{A}/(\iota'\Sigma\iota)$. Substituting for μ in equation (5), we find:

$$(\mathbf{w}\Sigma - \iota(\iota'\Sigma\iota)^{-1}\iota'\Sigma\mathbf{w}\Sigma)\mathbf{A} = \mathbf{A}\lambda \quad (6)$$

so that $\mathbf{A}$ is proportional to an eigenvector, and λ is the corresponding eigenvalue, of the matrix that premultiplies $\mathbf{A}$ on the left-hand side of (6). It is instructive to write the same equation in terms of $\mathbf{q}$:

$$(\Sigma - \Sigma\iota(\iota'\Sigma\iota)^{-1}\iota'\Sigma)\mathbf{wq}' \equiv \mathbf{M}'\Sigma\mathbf{Mwq}' = \mathbf{q}'\lambda \quad (7)$$

where $\mathbf{M} \equiv \mathbf{I} - \iota(\iota'\Sigma\iota)^{-1}\iota'\Sigma$. It will be recognized that the matrix $\mathbf{M}$ is the idempotent matrix such that $\mathbf{XM}$ is the vector whose jth element is the residual when the jth element of $\mathbf{X}$ (jth country's demeaned present value of income) is regressed on world present value of income. Thus, $\mathbf{M}'\Sigma\mathbf{M}$ (which equals $\Sigma\mathbf{M}$) is the variance matrix of residuals for each country, when each country's $\mathbf{X}$ is regressed on world present value of income and hence, if $\mathbf{w} = \mathbf{I}$, $\mathbf{q}'$ is (proportional to) an eigenvector of this matrix. Our world income component $\mathbf{D}$ is equal to $\mathbf{XA}$ (which equals $-\mathbf{X}\Sigma^{-1}\mathbf{q}'$); this is, if $\mathbf{w} = \mathbf{I}$, proportional to the first principal component of $\mathbf{XM}$, that is to $\mathbf{XMq}'$. To see this point, write $\mathbf{XMq}'$ as $\mathbf{X}\Sigma\mathbf{XMq}'$ and use the fact that $\Sigma\mathbf{Mq}' = \mathbf{q}'\lambda$.

Having solved the one-component case, let us now move to the general case. Disregarding, for the moment, the constraint that the $\mathbf{A}'\Sigma\mathbf{A}$ matrix should be diagonal, requiring only that its diagonal elements be one, we set up the Lagrangian:

$$L = tr(\mathbf{A}'\Sigma\mathbf{w}\Sigma\mathbf{A}) - \sum_{n=1}^{N}(\mathbf{A}_n'\Sigma\mathbf{A}_n - 1)\lambda_n - \iota'\Sigma\mathbf{A}\mu \quad (8)$$

where λ_n, $n = 1, \ldots, N$ are Lagrange multipliers for the constraint that the diagonal elements of $\mathbf{A}'\Sigma\mathbf{A}$ equal one, and where μ is the $N \times 1$ vector of Lagrange multipliers for the market clearing constraints. Differentiating with respect to the matrix $\mathbf{A}$, we find:

$$\frac{\partial L}{\partial \mathbf{A}} = 2\Sigma\mathbf{w}\Sigma\mathbf{A} - 2\Sigma\mathbf{A}\Lambda - \Sigma\iota\mu' = 0 \quad (9)$$

where Λ is a $N \times N$ diagonal matrix where the nth diagonal element is λ_n. Premultiplying (9) by $\mathbf{A}'$, we see that $\mathbf{A}'\Sigma\mathbf{A}\Lambda =$

$\mathbf{A}'\Sigma\mathbf{w}\Sigma\mathbf{A}$. Premultiplying (9) by ι', one finds that μ' equals $2\iota'\Sigma\mathbf{w}\Sigma\mathbf{A}/(\iota'\Sigma\iota)$. Substituting in (9) for μ', we then have:

$$(\mathbf{w}\Sigma - \iota(\iota'\Sigma\iota)^{-1}\iota'\Sigma\mathbf{w}\Sigma)\mathbf{A} = \mathbf{A}\Lambda \tag{10}$$

or, in terms of $\mathbf{q}$:

$$(\Sigma - \Sigma\iota(\iota'\Sigma\iota)^{-1}\iota'\Sigma)\mathbf{w}\mathbf{q}' \equiv \mathbf{M}'\Sigma\mathbf{M}\mathbf{w}\mathbf{q}' = \mathbf{q}'\Lambda \tag{11}$$

where $\mathbf{M} \equiv \mathbf{I} - \iota(\iota'\Sigma\iota)^{-1}\iota'\Sigma$. Premultiplying (11) by $\mathbf{w}^{.5}$, we see from the above expression that $\mathbf{w}^{.5}\mathbf{q}$ has columns proportional to eigenvectors of the real nonnegative definite symmetric matrix $\mathbf{w}^{.5}\mathbf{M}'\Sigma\mathbf{M}\mathbf{w}^{.5}$, and hence $\mathbf{q}\mathbf{w}\mathbf{q}'$ is diagonal. Using $\mathbf{q} = -\mathbf{A}'\Sigma$, we see that $\mathbf{A}'\Sigma\mathbf{w}\Sigma\mathbf{A}$ is also diagonal, and hence, using $\mathbf{A}'\Sigma\mathbf{A}\Lambda = \mathbf{A}'\Sigma\mathbf{w}\Sigma\mathbf{A}$, we note that $\mathbf{A}'\Sigma\mathbf{A}$ is the identity matrix, and we thus know that $\mathbf{A}'\Sigma\mathbf{w}\Sigma\mathbf{A} = \Lambda$. To maximize the trace of $\mathbf{A}'\Sigma\mathbf{w}\Sigma\mathbf{A}$ we select the N eigenvectors with the highest eigenvalues.

This contract design equation, (11), is the same as that in Theorem 8 of Athanasoulis and Shiller (2000) with the Γ matrix in theorem 8 equal to the identity. Thus, as we noted in the introduction, the pooling-WICs are the optimal premium-WICs conditioned on there already being a market for the world portfolio. They will have a zero premium since they represent only nonsystematic risk.

The matrices $\mathbf{A}$ and $\mathbf{q}$ are related by a couple of expressions. The matrix $\mathbf{A}$ equals $-\mathbf{M}\mathbf{w}\mathbf{q}'\Lambda^{-1}$. To see this, note that expression (11) is $\mathbf{M}\mathbf{w}\Sigma\mathbf{A} = \mathbf{A}\Lambda$, and use the fact that $\Sigma\mathbf{A} = \mathbf{q}'$. Hence, since $\mathbf{M}$ is idempotent, $\mathbf{M}\mathbf{A} = \mathbf{A}$. Let us define a $J \times J$ matrix $\mathbf{B}$ equal to $\mathbf{I} - \iota\iota'/J$. This is the matrix such that, for any vector $\mathbf{z}$, $\mathbf{B}\mathbf{z}$ is the vector $\mathbf{z}$ where the means have been subtracted, i.e. $\mathbf{B}\mathbf{z}$ is demeaned $\mathbf{z}$. Note that $\mathbf{B}$ is both idempotent and symmetric, with rank $J - 1$. Note also that $\mathbf{B}\mathbf{M} = \mathbf{B}$ and $\mathbf{M}\mathbf{B} = \mathbf{M}$. It follows that, if $\mathbf{w} = \mathbf{I}$ as in the empirical work below, $\mathbf{q}' = -\mathbf{B}\mathbf{A}\Lambda$, which means that columns of $\mathbf{q}'$ are the same as columns of $\mathbf{A}$, except that they are demeaned and rescaled by multiplying by minus the corresponding eigenvalue. To see that $\mathbf{q}' = -\mathbf{B}\mathbf{A}\Lambda$ in this case, note that since, when $\mathbf{w} = \mathbf{I}$, $\mathbf{A} = -\mathbf{M}\mathbf{q}'\Lambda$, $\mathbf{B}\mathbf{A} = -\mathbf{B}\mathbf{M}\mathbf{q}'\Lambda^{-1}$; since $\mathbf{B}\mathbf{M} = \mathbf{B}$, $\mathbf{B}\mathbf{A} = -\mathbf{B}\mathbf{q}'\Lambda^{-1}$. Using the fact that $\mathbf{B}\mathbf{q}' = \mathbf{q}'$, the result follows. Note also that $\mathbf{q} = \mathbf{q}\mathbf{B} = \mathbf{q}\mathbf{M}$. Because of these relations, we can write the income component security vectors in several different ways: $\mathbf{D} = \mathbf{X}\mathbf{A} = (\mathbf{X}\mathbf{M})\mathbf{A} = (\mathbf{X}\mathbf{M})\mathbf{B}\mathbf{A}$.

$\mathbf{I} + \mathbf{A}\mathbf{q}$ is the $J \times J$ matrix whose ijth element is the exposure, after hedging, of country j to country i's risk. If we include all possible components (that is, setting N equal to $J - 1$) so that the $(J - 1)$ eigenvectors of $\mathbf{w}^{.5}\mathbf{M}'\Sigma\mathbf{M}\mathbf{w}^{.5}$ are used to construct contracts, then, using (11), we see that $\mathbf{M}'\Sigma\mathbf{M} = \mathbf{q}'\mathbf{q}$. Then it can be shown that, regardless of the weighting matrix $\mathbf{w}$ chosen, $\mathbf{A}\mathbf{q} = -\mathbf{M}$ and $\mathbf{I} + \mathbf{A}\mathbf{q} = \iota\gamma$, where γ is the vector of regression coefficients when each country's present value of real income is regressed on the present value of world real income, that is $\gamma = (\iota'\Sigma\iota)^{-1}\iota'\Sigma$. That $\mathbf{I} + \mathbf{Aq} = \iota\gamma$ means that each country is holding a portfolio whose risk is the fitted value of its national income regressed on world income; everyone is completely diversified and subject to world income risk only. But such diversification does not generally occur unless we have $J - 1$ markets.

2.1. EXAMPLES

We have two examples below to help understand what the theory above does. The first example consists of two countries and shows what these contracts look like when one country is twice the size of the other in terms of variance. The second example shows a case with four countries where the first two countries are highly correlated but are uncorrelated with the second two countries. This will give us some intuition of what contract is chosen when certain countries are similar while others are dissimilar.

EXAMPLE 1:

In this example there are two countries one of which is riskier than the other and the country incomes are uncorrelated. Our Σ matrix is given by expression (12) below:

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{12}$$

Then, $\mathbf{M}'\Sigma\mathbf{M}$ has the form given by expression (13) below:

$$\mathbf{M}'\Sigma\mathbf{M} = \begin{bmatrix} 0{\cdot}67 & -0{\cdot}67 \\ -0{\cdot}67 & 0{\cdot}67 \end{bmatrix}. \tag{13}$$

which has one nonzero eigenvalue equal to 1·33. The vector $\mathbf{A}$, derived as shown above using the eigenvector corresponding to the largest eigenvalue is given by expression (14):

$$\mathbf{A} = 0{\cdot}82 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}. \tag{14}$$

Thus, the pooling-WIC may be described as just a short position in the first country and double an opposite long position in the other country. This contract is, as we might expect, a swap between the two countries but it is not a one-for-one swap. This occurs because we are constraining the countries to trade risk that has a zero premium. In contrast the premium-WIC, Athanasoulis and Shiller (2001), has equal and opposite positions in the two countries. But the market would not clear without payment of a premium since the first country would want to get rid of more of its own risk than the second country would be willing to take on. Thus the pooling-WIC weights the second country more than the first country. The vector **q** is given by expression (15):

$$\mathbf{q} = 0{\cdot}82[1 \quad -1]. \tag{15}$$

The first country is long in the pooling-WIC and the second country is short the pooling-WIC.

It is instructive to look at the matrix $\mathbf{I}+\mathbf{Aq}$ whose ith column gives the exposure of country i to risks in each of the two countries after hedging in the one market, expression (16).

$$\mathbf{I}+\mathbf{Aq} = \begin{bmatrix} 0{\cdot}67 & 0{\cdot}33 \\ 0{\cdot}67 & 0{\cdot}33 \end{bmatrix}. \tag{16}$$

Thus we have that the first country is exposed to $\frac{2}{3}$ of the risk of each country and the second country is exposed to $\frac{1}{3}$ of the risk in each country. This is as we would expect here. The two countries can reduce all of their nonsystematic risk and thus the first country is twice as large as the second and is exposed to $\frac{2}{3}$ of world risk while the second country is exposed to $\frac{1}{3}$ of world risk.

EXAMPLE 2:

In this example we study the case where there are four countries, and the first two countries are highly correlated with each other, but uncorrelated with the second two countries. Moreover, the second two countries are highly correlated with each other and all four countries have the same variance. Our Σ matrix is given by expression (17) below:

$$\Sigma = \begin{bmatrix} 1{\cdot}0 & 0{\cdot}9 & 0 & 0 \\ 0{\cdot}9 & 1{\cdot}0 & 0 & 0 \\ 0 & 0 & 1{\cdot}0 & 0{\cdot}9 \\ 0 & 0 & 0{\cdot}9 & 1{\cdot}0 \end{bmatrix}. \tag{17}$$

Then, $\mathbf{M}'\Sigma\mathbf{M}$ has the form given by expression (18) below:

$$\mathbf{M}'\Sigma\mathbf{M} = \begin{bmatrix} 0{\cdot}525 & 0{\cdot}425 & -0{\cdot}475 & -0{\cdot}475 \\ 0{\cdot}425 & 0{\cdot}525 & -0{\cdot}475 & -0{\cdot}475 \\ -0{\cdot}475 & -0{\cdot}475 & 0{\cdot}525 & 0{\cdot}425 \\ -0{\cdot}475 & -0{\cdot}475 & 0{\cdot}425 & 0{\cdot}525 \end{bmatrix} \tag{18}$$

which has one eigenvalue equal to 1·9 and two eigenvalues both equal to 0·1. The vector **A**, derived as shown above using the eigenvector corresponding to the largest eigenvalue is given by expression (19):

$$\mathbf{A} = 0{\cdot}36 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}. \tag{19}$$

Thus, except for scaling, the component may be described as just a short position in the first two countries and an equal and opposite long position in the other two. This contract is, as we might expect, a swap between the two blocks of countries. This component is quite different from the first principal component of Σ. That matrix has two first principal components, both with the same eigenvalue. These components are proportional to the vectors $[1\ 1\ 0\ 0]'$ and $[0\ 0\ 1\ 1]'$; if we created a market in either of these, then we would not provide any means for the two groups of countries to swap their risks. The vector **q** is given by expression (20):

$$\mathbf{q} = 0{\cdot}69[1\ 1 - 1 - 1]. \tag{20}$$

The first two countries are long in the pooling-WIC while the second two countries are short in the pooling-WIC.

If we were to create the next two markets, then each of these markets would entail a swap between the pairs of countries within one block. The risk reduction afforded by such swaps is much smaller because the countries are so highly correlated within each pair.

The matrix $\mathbf{I}+\mathbf{Aq}$ whose ith column gives the exposure of country i to risks in each of the four countries after hedging in the one market is as follows:

$$\mathbf{I}+\mathbf{Aq} = \begin{bmatrix} 0{\cdot}75 & -0{\cdot}25 & 0{\cdot}25 & 0{\cdot}25 \\ -0{\cdot}25 & 0{\cdot}75 & 0{\cdot}25 & 0{\cdot}25 \\ 0{\cdot}25 & 0{\cdot}25 & 0{\cdot}75 & -0{\cdot}25 \\ 0{\cdot}25 & 0{\cdot}25 & -0{\cdot}25 & 0{\cdot}75 \end{bmatrix}. \tag{21}$$

Not all elements of this matrix equal 0·25, as would be the case if we had included all three possible markets and thereby spanned the world risk-sharing opportunities, resulting in each country's holding one quarter of the world. Since we have only one market for trading income, it is not possible for each country to hold the world income, but the holdings shown in expression (21) do nearly as well for risk reduction, given the covariance matrix Σ that was assumed. For example, for country 1 the holding of 0·75 times its own income minus 0·25 times country 2's income is almost as good as the holding of 0·25 times its own income and 0·25 times country 2's income, given the high correlation between the two.

Suppose, to pursue this example further, that we changed the weight matrix **w** from the identity matrix to a matrix that gives much more weight to the first two countries, but keeping the weights constant within each country pair. This change in weights would have no effect on any of our optimal securities. Even if the contract designer cares primarily about the variance reduction of the first two countries, there is still nothing better that the market designer confined to one market can do for them than create a swap between this pair of countries and the other pair. And, if there is to be a second market, the best that can be done is to have a swap between the first two countries; if a third market, between the last two countries. If, on the other hand, the contract designer cares primarily about the first country, giving much more weight to it and equal weight to the other three countries, then the optimal first market will look very different; it will be approximately a swap between the first country and the rest of the world. Thus, giving unequal weight to countries that are in a grouping within which countries are highly correlated with each other can break the grouping up for contract definition.

2.2. WELFARE BENEFITS

We note, finally, that with the pure variance reduction method there is a convenient way of measuring the importance of each market. We can regress the jth country's national income on the nth index, (pooling-WIC), and take the variance of the fitted value in this simple regression, as the explained sum of squares for that country and market; this variance is just $\mathbf{q}_{nj}^2$. Since all the components are independent of each other, the sum of these variances $\sum_{n=1}^{N} \mathbf{q}_{nj}^2$ is the variance of the fitted value in a multiple regression on all of the components; if we add to this variance the variance of the residual in the regression, we get the total sum of squares, which is just $\mathrm{var}(\mathbf{X}_n)$. In our empirical work below we will show for each market, as a measure of its importance, the explained sum of squares as a percent of the total sum of squares. This measure gives a percent variance reduction measure of the welfare benefits.†

3. Data analysis

In Table 1 we report the largest and smallest ten-year growth rates of real *per capita* GDP for the ten largest countries for which we have GDP data 1950–1990. It is apparent that there is great variability among these growth rates for certain

† A similar measure is used in Athanasoulis and van Wincoop (2001) which uses the percent standard deviation reduction.

TABLE 1 *Highest and Lowest ten-year growth rates of real per capita GDP for countries with data for all years in the sample period*

Country	Date	Highest growth rate	Date	Lowest growth rate
Canada	1963–73	50·5%	1951–61	11·5%
Mexico	1971–81	53·5%	1980–90	–3·8%
USA	1958–68	35·8%	1973–83	6·3%
Brazil	1966–76	98·7%	1980–90	–6·1%
India	1979–89	47·6%	1964–74	–9·7%
Japan	1959–69	153·8%	1973–83	27·0%
France	1959–69	59·5%	1974–84	14·5%
Germany	1950–60	92·0%	1973–83	15·9%
Italy	1958–68	70·6%	1980–90	21·0%
UK	1963–73	33·0%	1973–83	10·6%

countries.† In Japan, the growth rates have varied from 27% to 154%. In Brazil, they have ranged from –6% to 99%. Plainly, changes in income of these magnitudes over ten-year intervals matter a lot to those receiving the income, and sharing the risk of such changes would have proven very beneficial to these people. These fluctuations in GDPs are very real; this is in contrast to the earthquakes or meteor impacts that theoretical economists often tell stories about, but which never appear in history to have caused economic dislocations that were remotely as big.

It is also apparent that the different countries have substantially different income growth paths through time, and that there is no simple shared pattern to the growth paths that would inspire the confidence that we know how to forecast them far out into the future. It is also apparent that there is a tendency for neighbouring countries to be substantially positively correlated with each other, and that distant countries may be uncorrelated or even negatively correlated with each other, not shown in the tables. Some correlations are estimated to be negative: India and Japan happen to show large negative correlation over this period. Because there is not much information about correlations in these data, which are dominated by low-frequency movements and for which we have no secure model, we cannot attach much confidence that national incomes in these countries really tend to move opposite each other.

For our analysis, we must convert these impressions into estimates of the matrices Σ. Estimating the variance matrices is not a trivial matter; these are supposed to reflect the *conditional* variance at the time of the contract for distant future national incomes. To estimate such a variance matrix, we need first

† We report variances for the present value of national incomes in Table 3.

to form some representation of the conditional expected value each year for all future national incomes, a problem that the world's macroeconomic forecasters have been spending decades to develop.

There are many models that might be used to provide estimates of Σ. Estimating time series models, such as autoregressive models, for the national income of each country would help us to separate out which components of national incomes are forecastable and which are not. There is, however, a risk inherent in specifying any simple autoregressive model; that it will not capture accurately the long-term risks that we want to hedge. Estimating spatial models, such as the spatial autoregressive models or other Markov random field models, would allow us to put structure on the matrix Σ so that fewer parameters would be estimated, so that our shortage of information about long-run risks would present less of an estimation problem. Spatial models could use sophisticated concepts of economic distance between countries, or prior information about the similarity of different countries. There is a risk in any spatial model specification, though, that we may be using the wrong measure of economic distance between countries, and therefore impose incorrect priors or restrictions on our variance matrices.

It is beyond the scope of this paper to set forth a definitive treatise on how to estimate Σ; we leave that for possible future work. For this paper we used two very simple methods to estimate, methods that appear to be transparent and fairly robust to many kinds of possible misspecifications, with the hope that our estimates will be at least suggestive of the new markets that may be created. Our methods of producing the Σ matrices will at least capture in some fashion the magnitude of variability to be idiosyncratic, and the tendency for some measure of comovement across countries, even if the estimated matrices are not highly accurate. At this stage in our research, we approach the problem in almost the same spirit that real business cycle modelers who "calibrate" their models have. We are hoping to tell a simple story that has an important element of truth in it, and are not now particularly interested in testing our variance matrix model against general alternatives; even if the model were rejected the estimated Σ may yet be useful for our purposes.

Our two methods of estimating Σ differ in what they assume about the representativeness of past historical movements for the future. Our Method A, which involves estimating simple unconstrained variance matrices from historical data, makes no assumptions about similarities of, or economic distances between countries. This method, since it requires a lot of data, is used only for a rather low T, equal to ten years; even with this low T, we do not expect to get accurate estimates of variances.† Our Method B, which involves estimating constrained variance matrices, imposes some strong priors and thereby saves degrees of freedom so that we have better prospects of estimating variance matrices with high T; with Method B we use T equal to forty years. Neither method makes use of time series models to infer conditional moments; both are based on the assumption that conditional variances of long-horizon changes in income are best estimated directly as moments of long-horizon changes themselves. Our motivation is the notion, based on our reading of others' success in forecasting, that ten-year or forty-year changes in national incomes are virtually unforecastable.

3.1. METHOD A

To estimate Σ we take the sample variance matrix for the J countries of $\mathrm{GDP}_{1990} \times \sum_{i=1}^{T} \mathrm{gdp}_{t+i}/((1+\rho)^i \mathrm{gdp}_t)$. Our sample period is $t = 1950, \ldots, 1990 - T$, with 41-T observations where GDP denotes total, not *per capita*, gross domestic product in 1990 (in 1985 dollars), and gdp_t denotes real *per capita* gross domestic product in year t (in 1985 dollars).

3.2. METHOD B

With our second method of estimating Σ, we impose prior restrictions that all countries have the same mean and variance of percentage changes in real *per capita* income, and that covariances are determined solely by the geographic distances between countries.‡ The motivation for requiring that all countries have the same mean and variance of percentage changes of real *per capita* income is some skepticism that the past exigencies that faced particular countries during 1950–1990 can really be expected to repeat in those same countries in the future. Our data show that Japan has had much higher growth rates than most

† We have only four nonoverlapping time intervals with which to compute variances of ten-year present values. Supposing that the variables are normal and independent across the four time intervals, and approximating our variance as estimated from four such observations, then the variance estimate will be proportional to a χ^2 variate with three degrees of freedom, and an 80% confidence interval for a standard deviation from 80% of the estimated standard deviation to 262% of the estimated standard deviation. We have not tried to produce standard errors for our variance matrices, since such standard errors would depend on the assumed model for our processes and there are many possible models to which we at this point attach prior probability. Further refinement of our knowledge about Σ is left to later work.

‡ A similar method is used in Athanasoulis and Shiller (2001).

of the other countries. Do we really have reason to expect that growth rates will be similarly higher in the future in Japan? Our Table 1 suggests that Japan and Brazil are risky countries. Do we really have any reason to think that these countries will be the ones facing the greatest risks in the future? Perhaps they are just buffeted by some major crises in this sample, crises the likes of which may just as well strike other countries in the future. The motivation for requiring that the correlation across countries in percent changes in real *per capita* income depends only on the distance between the countries is much the same, we do not really attach much credence to the suggestion of simple variance matrices computed by Method A that India and Japan should be expected to be negatively correlated in the future.

Our prior assumptions for Method B about the variance matrix **V** of T-year percentage changes in real *per capita* national incomes are represented by the formula:

$$\ln(\mathbf{V}_{ij}) = a - bd_{ij} \quad b \geq 0 \tag{22}$$

where d_{ij} is the distance between countries i and j, measured as air miles between the major cities in the respective countries. We used the air mile distances between the major cities Montreal, Mexico City, New York, Rio de Janeiro, Calcutta, Tokyo, Paris, Berlin, Rome, London, Shanghai and Moscow. Since b is positive, the further away the major city, the less is the covariance with its country. This formula corresponds to a valid (i.e. the variance matrix is nonnegative definite for any placement of cities) isotropic (i.e. the model is invariant to rotations of the coordinate system) spatial model where the cities lie in R^2, see Cressie (1991, p. 86). The formula also corresponds to a valid isotropic spatial model where the cities lie on the surface of a sphere and distances are measured along great circles, as in our application to the earth. Moreover, the variance matrix is strictly positive definite unless two cities coincide.

This formulation restricts all covariances to be positive. The prior restriction that all covariances are positive may seem strong, but it is maintained here as a sort of commonsense prior notion that there is really no reason in general for any pairs of countries to tend to move opposite each other. This restriction may serve to reduce the possibilities for diversification, by eliminating the negative correlations that diversifiers seek.

Our assumption that the variance of the percentage change in incomes is the same in all countries reflects an underlying assumption that people within countries share a country risk common to them all, so that national borders have some economic significance, and are not just random closed curves on a map. If national borders had no economic significance, then we might expect that larger countries would have smaller variances of percentage changes, since there would be more opportunity for geographical diversification within the larger countries. This assumption of constant percentage change variances will be important to some of our results, since it implies that diagonal elements of Σ are related to x_{0j}^2 rather than to x_{0j}, where x_{0j} is initial year national income for country j.

We compute the constrained maximum likelihood (multivariate normal) variance matrix $\mathbf{V}$ for the 10 countries of $z_j = \sum_{i=1}^{T} \text{gdp}_{1990}/(\text{gdp}_{1990-i}(1+\rho)^i)$, $j = 1, \ldots, 10$; there is only one observation of z_j for each country, but there are only three unknown parameters, a, b and the mean growth rate. Then using distance data and real national income data for China and the CIS (the latter including the Baltic countries, so that it corresponds to the former Soviet Union), we construct using (22) a twelve by twelve $\mathbf{V}$ matrix for the twelve countries, and using the Summers-Heston data for real GDP for all countries in 1990, we construct a twelve by twelve $\mathbf{X}_0$ matrix for all twelve countries, where x_{0j} is the jth diagonal element. Then we take $\Sigma = \mathbf{X}_0\mathbf{V}\mathbf{X}_0$.

Note that the maximum likelihood method will tend to produce downwardly biased estimates of variance, since with only one observation for each country, the estimated mean will tend to pick up the component of the variation that is shared by all countries; recall that in the iid case the maximum likelihood estimate of variance is the sum of squared residuals divided by N rather than $N-1$. The downward bias will be more severe here than in the iid case, since our countries are positively correlated with each other by assumption. Still, the maximum likelihood estimate is the posterior mode based on uninformative priors, and we think that this conservative estimate of variance is acceptable for our purposes.

Our method also requires us to specify a weighting matrix $\mathbf{w}$ and a discount rate ρ for our problem that defines the contract weights that are represented in the matrix $\mathbf{A}$. We choose a weight matrix equal to the identity, $\mathbf{w} = \mathbf{I}$. With this choice of the weight matrix we are minimizing total variance. This weighting matrix preserves a simple correspondence between our method and principal components analysis. We choose a discount rate of $\rho = 2\%$.

4. Results

We present results with $N = 2$, two markets and the parameter values described above though we also present results on the total percent variance reduction available if all contracts were constructed. In Table 4 the results are presented first for variance matrix estimation method A (unconstrained) ten countries, and in

Table 5 for variance matrix estimation method B (constrained) and twelve countries.

In Table 2 we report the 1990 GDPs of countries, in 1985 dollars, along with their world shares of GDP and their populations. For our ten-country case the two largest countries are the U.S. and Japan. In the twelve country case the largest countries are the U.S. and China followed by Japan and the CIS. It would seem that the largest countries should be the ones that dominate the first two contracts though as we will see this is not necessarily the case. If the riskiness of each country's GDP were proportional to their size, and all countries had the same percentage of systematic and nonsystematic risk then this would be the case. However, imagine a very large country which is very risky but all of its risk is systematic risk; then this country would not be important in any of these contracts.

The estimated Σ matrices (not shown), whether constrained or unconstrained, show that near neighbours tend to have higher correlations than do more distant countries.† With the unconstrained variance matrices, covariances are usually positive with the exception of India, whose covariance is estimated to be negative with most other countries. With the constrained variance matrix estimates, all covariances are constrained to be positive. For the constrained variance matrix estimate, corresponding to Table 5, the estimated correlation between (forty-year present values of income in) the U.S. and Canada is 0·88, between France and Germany is 0·80, between China and Japan is 0·65. The estimated correlation between distant countries is quite small: the correlation

TABLE 2 *GDP, world share in output and populations in 1990*

Country	1990 GDP in 1985 dollars $\times 10^{11}$	World Share in %	Population $\times 10^{7}$
Canada	4·57	2·83	2·62
Mexico	4·55	2·82	8·46
U.S.	45·8	28·38	24·9
Brazil	5·76	3·57	14·7
India	8·89	5·51	83·3
Japan	18·3	11·32	12·3
France	7·83	4·85	5·62
Germany	8·92	5·53	6·15
Italy	7·23	4·48	5·75
U.S.	7·48	4·64	5·72
China	26·3	16·33	113
CIS	15·7	9·73	28·8
Sum	161	100	312

† An appendix showing detailed results is available from the authors.

between the U.S. and Japan is 0·07, between the U.S. and the CIS is 0·16, between the CIS and China is 0·19. India of course no longer has negative correlation with anyone: its correlation with China is estimated at 0·43, with Japan, 0·28, with the U.S., 0·04.

In choosing these pooling-WICs to minimize variance across countries, it is not the variance of the present values of incomes that is important but rather the variance of the residual of the present value of country incomes regressed on the present value of world income. It is nonsystematic risk that matters. To get an idea of the amount of variance which is nonsystematic in the present value of country incomes, we report some of these numbers in Table 3 for both the unconstrained case and the constrained case. For the ten-country cases we see that in terms of variance, Japan is the largest country even though it was not the largest in terms of 1990 GDP. The U.S. is the second largest in terms of variance. Even though this is the case, only 10% of Japan's variance is nonsystematic while approximately 50% of the U.S.'s variance is nonsystematic. Thus the U.S. has more total nonsystematic risk than Japan even though the U.S. has about one-third of Japan's total variance. It is the total nonsystematic variance that matters. In the unconstrained case the largest countries in terms of nonsystematic variance are from largest to smallest the U.S., Japan, Brazil and Germany. Note that in terms of GDP, Brazil is a small country. We see in the constrained case that in terms of nonsystematic variance the largest countries are the U.S., China, Japan and the CIS.

We can make some general observations about these contracts in terms of the contract weights, that is, in terms of the columns of **A**. In Table 4, the first contract can be described in rough terms as approximately a swap between the U.S. and the Far East (in Table 4, the Far East is represented only by Japan, in Table 5 by Japan and China), though in Table 4 Germany also plays an important role. The reason for this can be found in the correlation coefficients for the residuals when regressing the present value of country income on the present value of world income, i.e. correlation coefficients of residual risk of country incomes. The correlation coefficient between the nonsystematic risk of the U.S. and Japan is −0·37, between the U.S. and Germany is −0·76 and between Germany and Japan is 0·12. Thus while Germany has less nonsystematic risk than Japan, see Table 3, Germany's nonsystematic risk is more negatively correlated with that of the U.S. than Japan's is with the U.S. and thus there is much risk reduction between the U.S. and Germany. The second market in Table 4 is basically a Brazil-Japan swap of risk. Both of these countries have much nonsystematic risk which can be diversified and the correlation coefficient of their nonsystematic component of income is −0·66 so there are large variance reduction opportunities.

TABLE 3 *Variance of the present value of GDPs and the total % of variance which is nonsystematic and total variance which is nonsystematic for the unconstrained variance matrix and the constrained variance matrix*

Country	Unconst. Var. $\times 10^{22}$	% nonsys. Var.	Unconst. nonsys. Var. $\times 10^{22}$	Const. Var. $\times 10^{25}$	% nonsys. Var.	Const. nonsys. Var. $\times 10^{25}$
Canada	7·16	88·02	6·30	3·51	56·44	1·98
Mexico	7·25	68·18	4·94	3·48	86·81	3·02
U.S.	379	49·86	189	352	50·72	179
Brazil	64·9	93·45	60·7	5·57	96·04	5·35
India	26·5	65·32	17·3	13·3	84·11	11·2
Japan	987	10·01	98·8	56·1	74·12	41·6
France	34·3	6·01	2·06	10·3	65·37	6·73
Germany	84·2	68·40	57·6	13·4	64·29	8·61
Italy	29·2	27·44	8·01	8·78	71·05	6·24
U.K.	4·32	73·99	3·20	9·41	64·94	6·11
China				117	69·32	81·1
CIS				41·4	67·63	28·0

TABLE 4 *Unconstrained variance matrix, ten countries, 10-year contracts*

Country	$\mathbf{A}_{j1}$ weights $\times 10^{-10}$	$\mathbf{q}_{1j}$ positions $\times 10^{9}$	ESS/ TSS benefit % of σ^2	$\mathbf{A}_{j2}$ weights $\times 10^{-10}$	$\mathbf{q}_{2j}$ positions $\times 10^{8}$	ESS/ TSS benefit % of σ^2	ESS/ TSS total benefit % of σ^2
Canada	0·82	−0·20	55·00	4·01	−1·09	16·54	88·02
Mexico	0·22	−0·04	2·60	3·97	−1·04	14·78	68·18
U.S.	5·26	−1·34	47·42	1·06	2·76	2·01	49·86
Brazil	0·05	−0·00	0·00	8·68	−7·18	79·51	93·45
India	−0·75	0·21	15·99	3·01	0·21	0·17	65·32
Japan	−1·98	0·52	2·75	−3·05	8·12	6·68	10·01
France	−0·30	0·09	2·27	3·15	0·03	0·00	6·01
Germany	−2·29	0·60	42·90	4·40	−1·60	3·06	68·40
Italy	−0·82	0·22	16·85	2·90	0·36	0·44	27·44
U.K.	0·26	−0·05	6·81	3·60	−0·56	7·25	73·99

Notes: Weights $\mathbf{A}_{jn}$, $n = 1, 2$, is fraction of country j detrended GDP paid as part of dividend on one security n; position $\mathbf{q}_{nj}$, $n = 1, 2$, is total number of securities n the theory predicts will be owned in country j; ESS/TSS is explained sum of squares over total sum of squares. The last column is the total ESS/TSS if all nine contracts were constructed. Source: Calculations by authors using data 1950−1990 form Summers and Heston (1991); see text.

In Table 5, where we have added China and the CIS, the first market may be described as a U.S.-rest of the world swap with a

TABLE 5 *Constrained variance matrix, twelve countries, 40-year contracts*

Country	$\mathbf{A}_{j1}$ weights $\times 10^{-11}$	$\mathbf{q}_{1j}$ positions $\times 10^{10}$	ESS/TSS benefit % of σ^2	$\mathbf{A}_{j2}$ weights $\times 10^{-11}$	$\mathbf{q}_{2j}$ positions $\times 10^{10}$	ESS/TSS benefit % of σ^2	ESS/TSS total benefit % of σ^2
Canada	−0·24	−0·35	34·13	0·50	0·06	0·94	56·44
Mexico	−0·32	−0·16	7·20	0·54	0·03	0·25	86·81
U.S.	1·38	−4·12	48·20	−0·60	0·91	2·37	50·72
Brazil	−0·38	−0·04	0·29	0·65	−0·06	0·61	96·04
India	−0·56	0·39	11·64	0·53	0·04	0·10	84·11
Japan	−0·90	1·18	24·74	−0·63	0·94	15·61	74·12
France	−0·44	0·12	1·30	1·35	−0·60	34·90	65·37
Germany	−0·48	0·19	2·76	1·51	−0·73	39·98	64·29
Italy	−0·45	0·14	2·20	1·27	−0·54	32·81	71·05
U.K.	−0·43	0·10	0·97	1·29	−0·56	32·89	64·94
China	−1·28	2·06	36·51	−1·67	1·75	26·13	69·32
CIS	−0·60	0·49	5·72	2·16	−1·24	36·93	67·63

Notes: Weights $\mathbf{A}_{jn}$, $n = 1, 2$, is fraction of country j detrended GDP paid as part of dividend on one security n; position $\mathbf{q}_{nj}$, $n = 1, 2$, is total number of securities n the theory predicts will be owned in country j; ESS/TSS is explained sum of squares over total sum of squares. The last column is the total ESS/TSS if all eleven contracts were constructed. Source: Calculations by authors using data 1950−1990 form Summers and Heston (1991); see text.

lot of weight on the CIS. The second market might be described in simple terms as a swap between the European community and CIS on one side, and China, Japan and the U.S. on the other. Note that in these contracts as well as the ones for Table 4, it is the largest countries in terms of nonsystematic variance which dominate the contracts since the contracts are chosen to minimize total variance. It is the size and the correlation coefficient of nonsystematic income risk which matter for contract design.

To help understand the explained sum of squares over total sum of squares, consider the first market of Table 4. The explained sum of squares over total sum of squares is 47·42% for the U.S., indicating that this one security makes it possible for the U.S. to get rid of nearly half of its uncertainty about income and exhausts nearly all of its residual risk-sharing opportunities, see the last column in Table 4. It offers much less benefit for Japan as a percent of its variance. Japan derives less benefit since its GDP variance is estimated to be three times higher than that of the next highest country, the U.S., and its income is substantially positively correlated with the U.S.; there is in essence no one who benefits from taking on much of the Japanese risk. Japan has essentially exhausted most of its opportunities to lay off income risk; no one else has much risk to swap relative to Japan's. Brazil benefits enormously from the second contract. Much of the risk-sharing opportunities for the U.S., Japan, Brazil and Germany are exhausted by the first two pooling-WICs in the unconstrained variance case.

In Table 5 we see that a large portion of the risk-sharing opportunities are exhausted for the U.S., and China. Japan, the CIS and the E.U. (Germany, France, Italy and the U.K.) all have about equal risk reduction in the first two contracts. The reason is that the two largest countries in terms of diversifiable risk are the U.S. and China. While Japan is the next largest it must be remembered that by proximity, the E.U. almost forms a single country so that the E.U. as a group is very important for risk sharing and thus we see that the other risk sharing benefits do not only accrue to Japan and the CIS but also to the E.U.

Thus the weights in the pooling-WICs are driven by a combination of the size in terms of variance, the percent of that variance which can be hedged and the correlation coefficients of the residual risk. The most important countries will be those which have the largest amount of variance that can be hedged.

5. Conclusion

We believe that CLIC securities should be created that somewhat resemble the pooling-WIC contracts defined in either Table 4 or, preferably, Table 5 which was estimated with common-sense priors. Since we have argued before (2000) that there should be a market for the world portfolio, it follows from above that the next markets should pooling-WIC contracts. We have seen that in Table 5 the first pooling-WIC contract is roughly a swap between the U.S. on one side and the rest of the world on the other. The second is a swap between greater-Europe (including the CIS) on the one side and the U.S. and Far East on the other. The specification of these pooling-WIC contracts needs to be explored further with more research on estimation of variance matrices.

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