

## ACTUAL AND WARRANTED RELATIONS BETWEEN ASSET PRICES

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### 1. Introduction

A VARIETY of efficient markets models can be represented in the form  $P_{it} = E_t P_{it}^*$  where  $P_{it}$  is the price of asset  $i$  at time  $t$  and  $P_{it}^*$  is its *ex post* value, i.e., fundamental value.<sup>1</sup> In this paper, we inquire what such models imply for the covariance and for the correlation between the prices of the assets in terms of the covariance matrices of the *ex post* values. We show that while knowledge of the covariance matrix between *ex post* values does not permit us to predict, using efficient markets models, the covariance between prices, it does allow us to put bounds on this covariance.

Certainly, there is a common-sense presumption that these price covariances, correlations, as well as betas and factor loadings, depend on the covariances or correlations of *ex post* values. If an observed covariance between prices is to be justified in terms of these models, then there must be enough covariance between *ex post* values to warrant the covariance between prices. But apparently the limits on the actual covariances between prices that may be warranted by the covariance matrix of *ex post* values have never been set forth in a general form before. These are important limits to set forth, since empirical finance is widely concerned with observed correlations among asset prices, and much work is based on the general notion that these have something to do with fundamentals. We will apply our theory to a study of the covariance and correlation of log price–dividend ratios between the United Kingdom and the United States.<sup>2</sup> In so doing, we will be able to offer some evidence on the claims of some who, observing the UK and US stock markets often rise and crash together, have doubted that information about fundamentals in the two countries could justify the extent of the co-movements.

Knowing the relations among these covariances and correlations is important for a number of purposes. They would help us to understand whether international transmission of asset price movements must be understood in terms of something other than the simple present value models; that is our immediate objective here. Beyond that, they may help us to understand how fundamentals interact with investor information to determine betas or factor loadings of asset prices.

<sup>1</sup> Typically, *ex post* value is a present value of dividends per share at time  $t$ . It may have other interpretations as well; for example if  $P_t$  is a forward price *ex post* value could be the subsequent spot price. Also, prices and *ex post* values may be transformed as in the empirical work below.

<sup>2</sup> The transformation of price and of the present value referred to above is its log minus the log dividend. This is a nonlinear transformation, but the transformation can be justified in terms of an approximation to the present value model; see Campbell and Shiller (1988). The transformation causes the variable  $P_{it}$  to be stationary through time.

Many empirical studies of relations among financial prices refer either explicitly or implicitly to covariances among fundamentals to motivate the construction of the study or the interpretation of results. For example, Fama and French (1989, pp. 3–4) examine whether forecasting variables related to business conditions track common variation of expected returns on bonds and stocks for the US 1927–87. They ask: ‘Are the relations [between returns] consistent with intuition, theory and existing evidence on the exposure of different assets to changes in business conditions?’ and they conclude that the results are ‘comforting.’ Chen *et al.* (1986, p. 402), in their study of economic forces in the US stock market 1953–84 and in their selection of macroeconomic factors to use for stock market returns appear to base their selection on the likely correlation of fundamental with these. They write: ‘Our conclusion is that stock returns are exposed to systematic economic news, that they are priced in accordance with their exposures, and that the news can be measured as innovations in state variables whose identification can be accomplished through simple and intuitive financial theory.’ Pindyck and Rotemberg (1988, 1990, p. i) estimate multiple indicator multiple cause (MIMIC) models of asset prices where the indicators are returns and the causes are economic variables related to fundamentals. They conclude from their study of US stock prices 1969–87 that ‘we show that comovements of individual stock prices cannot be justified by economic fundamentals’. King *et al.* (1990, p. i) estimate a factor model for 16 national stock markets 1970–88 using ten macroeconomic variables. They conclude that ‘the main empirical finding is that only a small proportion of the time-variation in the covariances between national stock markets can be accounted for by observable economic variables’. The conclusions in these different studies seem rather varied; it is worth examining whether these claims can be evaluated in terms of a consistent and rigorous theoretical framework.

The key problem in carrying out the objective of reconciling correlations of prices with correlations of fundamentals is that we do not observe the full information set available to market participants to forecast present values, and in the framework of efficient markets theory, we must assume that market participants might have superior information. This means we cannot observe the optimal forecast at time  $t$  of  $P_{it}^*$ , cannot observe directly its covariance with anything, and thus cannot calculate just what the covariance of prices should be.

We can only put bounds on the warranted covariances of prices from the knowledge of variance matrices of *ex post* values. In Section 2 we derive covariance bounds for the case when no forecasting information is available for statistical analysis, while in Section 3 we show that using more information is helpful both in deriving more efficient covariance bounds and in deriving bounds for the warranted correlation between the two assets. Section 4 contains a description of the pricing theories that we use for stocks, Section 5 describes the data. In contrast to most of the aforementioned studies, we use very long time series data, extending from 1919 to the present, to enable us to see more of the low frequency variation in fundamentals that might explain covariances between prices, and in contrast to all of these studies we use direct measures

of *ex post* or fundamental value. Section 6 describes the econometric methodology, and Section 7 gives the results.

## 2. The case of no forecasting information available for statistical analysis

Suppose first that we wish to base our statistical analysis only on the covariance matrix of the vector  $P_t^* = [P_{1t}^*, P_{2t}^*]'$ , whose  $i$ th element is the present value of the dividends accruing to asset  $i$ . The corresponding vector of prices  $P_t$  has as its  $i$ th element the price of asset  $i$ . By basing our analysis only on these covariance matrices, we are attempting to see in very simple and basic terms whether we can find evidence of excessive co-movement of prices. We will suppose that the present values and corresponding prices have been suitably transformed so that they are stationary, and so that variance matrices  $\text{var}(P^*)$  and  $\text{var}(P)$  exist.

How large can the covariance between  $P_{1t}$  and  $P_{2t}$  be, given  $\text{var}(P^*)$ ? That is, knowing how much the fundamental variables vary and co-move, how much can  $P_{1t}$  and  $P_{2t}$  co-move? To answer this, we must solve a nonlinear programming problem: maximize  $\text{cov}(P_{1t}, P_{2t})$  with respect to  $\sigma(P_1)$  and  $\sigma(P_2)$  subject to the inequality restrictions implicit in the requirement that the  $2 \times 2$  matrices  $\text{var}(P)$  and  $(\text{var}(P^*) - \text{var}(P))$  are both positive semidefinite.<sup>3</sup> This means that there are eight inequality constraints: the four variances must be positive and (from the restrictions that the determinants must be positive) there are upper and lower bounds to the two covariances. Since we are maximizing, only the upper bounds to the covariances could be binding. The variance constraints mean that, in  $\sigma(P_1), \sigma(P_2)$  space, the point  $(\sigma(P_1), \sigma(P_2))$  must lie in a box in the positive quadrant from 0 to  $\sigma(P_1^*)$  and from 0 to  $\sigma(P_2^*)$ . The remaining two constraints are

$$\text{cov}(P_1, P_2) \leq \sigma(P_1)\sigma(P_2) \quad (1)$$

$$\text{cov}(P_1, P_2) \leq \text{cov}(P_1^*, P_2^*) + ((\sigma(P_1^*)^2 - \sigma(P_1)^2)(\sigma(P_2^*)^2 - \sigma(P_2)^2))^{.5} \quad (2)$$

Isoquants for the constraint on  $\text{cov}(P_1, P_2)$  in (1) are rectangular hyperbolas within the box in  $\sigma(P_1), \sigma(P_2)$  space, so that there are asymptotes to both abscissa and ordinate. Higher values of the covariance permitted by (1) occur as we move from one isoquant to isoquants positioned higher up or to the right. Isoquants for the constraint on  $\text{cov}(P_1, P_2)$  in (2) are also negatively sloped throughout the box, but are concave down instead of up. Higher values of the covariance permitted by (2) occur as we move from one isoquant to isoquants positioned down and to the left. Tangencies between these sets of isoquants lie along a diagonal straight line across opposite corners of the box, the straight line from  $(0, 0)$  to  $(\sigma(P_1^*), \sigma(P_2^*))$ . Thus, the maximum will be found at one of these tangencies, along the straight line, where both constraints (1) and (2) will

<sup>3</sup> Here, the term positive-semidefinite is taken to allow strictly positive definite matrices as well as singular ones.

be binding at the maximum, and none of the other constraints will be binding.<sup>4</sup> Since these are the only two constraints that bind, we can find the upper bound to the covariance by solving the simple problem of maximizing  $\sigma(P_1)\sigma(P_2)$  with respect to  $\sigma(P_1)$  and  $\sigma(P_2)$  subject to the constraint that the difference between the right hand sides of (1) and (2) equals zero. Solving this problem, and the corresponding minimization problem, we find bounds on the covariance:

$$(\text{cov}(P_1^*, P_2^*) - \sigma(P_1^*)\sigma(P_2^*))/2 \leq \text{cov}(P_1, P_2) \leq (\text{cov}(P_1^*, P_2^*) + \sigma(P_1^*)\sigma(P_2^*))/2 \quad (3)$$

We shall refer to the range specified in this inequality as the range of warranted covariance between  $P_1$  and  $P_2$ . Note that the warranted covariance between prices can exceed the covariance between the present values, even when this covariance is positive. As noted in Shiller (1989), this happens when there is 'positive information pooling,' when the forecast error  $P_{1t}^* - P_{1t}$  is negatively correlated with the forecast error  $P_{2t}^* - P_{2t}$ . In this case, the variance of the forecast error  $P_{1t}^* + P_{2t}^* - (P_{1t} + P_{2t})$  is less than the sum of the variances of the individual forecast errors. In this case, information is more about the aggregate  $P_{1t}^* + P_{2t}^*$  than about the individual present values, i.e., the information about the present values is pooled.

Note, for example, from (3), that if  $P_{1t}^*$  and  $P_{2t}^*$  are highly positively correlated, then  $P_{1t}$  and  $P_{2t}$  can have both positive or negative covariance, but possible covariances include large positive covariances but only small negative covariances. For another example, note that if  $P_{1t}^*$  and  $P_{2t}^*$  are uncorrelated and have the same variance, then the covariance between  $P_{1t}$  and  $P_{2t}$  can range between minus half the variance to plus half the variance.

It was concluded in Shiller (1989) that, for a transformation of UK and US stock prices indexes 1918–88 (where the transformation consists of dividing price by a long moving average of lagged dividends)  $\text{cov}(P_1, P_2)$  exceeded  $\text{cov}(P_1^*, P_2^*)$  with a constant discount rate used to compute present values, and there was no evidence of information pooling. It is not surprising, therefore, that the inequality (3) is strongly violated with that data too. However, when the discount rate is allowed to vary with the prime commercial paper rate, the bounds in (3) are no longer violated.<sup>5</sup>

That result, if valid, implies that with constant discount rates there is excess covariance between the UK and the US stock prices. But, it does not tell us whether or not there is excess correlation between the two countries' stock prices. Covariance tells us the magnitude of their co-movements, but does not tell us whether the two prices closely resemble each other.

<sup>4</sup> However, in the special case where  $P_1^*$  and  $P_2^*$  are perfectly correlated, the maximum will occur on the diagonal at the upper right corner of the box.

<sup>5</sup> Variance matrices  $\text{var}(P)$  and  $\text{var}(P^*)$  are given in Table 1 of Shiller (1989). The covariance between  $P_{1t}$  and  $P_{2t}$  1919–87 (between the transformed UK and US prices) was reported as 39.73. With a constant discount rate assumption, the upper bound allowed by (1) using the estimated covariance matrices is 8.84. With a discount rate varying with the prime commercial paper rate, the upper bound allowed by (1) using the estimated covariance matrices is 45.18.

In fact, if we have only  $\text{var}(P^*)$  to work with, lacking any components of the information set that the public uses to forecast, and if this matrix is not singular, then we cannot say anything at all about the warranted correlation between  $P_1$  and  $P_2$ . As long as  $\text{var}(P_t^*)$  is nonsingular we can always write  $P_t^* = u_t + v_t$  where  $u_t$  and  $v_t$  are random vectors uncorrelated with each other, and  $v_{1t}$  and  $v_{2t}$  are uncorrelated with each other, and all elements have non-zero variances. Suppose that information consists of  $u_{1t} + u_{2t} + \text{noise}$ ,  $v_{1t}$  and  $v_{2t}$ . As the variance of  $v_{1t}$  and  $v_{2t}$  are taken to zero, the correlation between prices approaches 1.00. As the variance of noise is increased toward infinity, the correlation approaches zero.

Now suppose that the second asset is the return on the market portfolio, that prices are scaled to 1.00 in the preceding period, that there are no dividends paid this period and that the variance matrix of  $P^*$  is conditional on information before this period. Then the conditional beta of the first asset is given by  $\beta = \text{cov}(P_1, P_2)/\text{var}(P_2)$ . We can always write  $P_t^* = u_t + v_t$  where  $u_t$  and  $v_t$  are random vectors uncorrelated with each other, and as long as  $\text{var}(P^*)$  is strictly positive definite, we can take the  $\text{var}(u_1)/\text{var}(u_2) = x$  for arbitrary positive  $x$ . Suppose the information set consists only of  $u_1 + u_2$ . Then the beta is  $x$ , which can be made anything from 0 to infinite for positive  $x$ . It can similarly be shown that beta can also range from 0 to minus infinity by taking the information vector to be  $u_1 - u_2$ . Thus, the variance matrix of fundamental values places no restriction at all on beta. It is still possible to put bounds on the correlation between the two prices, or on the beta of an asset, even without specifying the full information set used by market participants, so long as we know part of the information set used by the market. Using such a subset of public information also allows us to tighten our bounds on the covariance between  $P_{1t}$  and  $P_{2t}$ .

### 3. The case when forecasting information is available for analysis

If we know a subset of the information set available by market participants to forecast present values, and thereby observe the variance matrix of the forecast  $P'_t = E(P_t^* | I_t)$  where  $I$  is the subset of information, then this will allow us to put tighter bounds on the warranted covariance between prices. We assume that only a subset, and not the whole set, of the information used by market participants is available for statistical analysis, since much information that participants used has not been reduced to quantified time series.

We can, following Campbell and Shiller (1988a, b), include the vector of actual prices in the subset of information, since surely the market knows market prices. Under the efficient market hypothesis, then  $P'_t$  should equal  $P_t$ , and so under the efficient markets hypothesis the covariance between  $P'_{1t}$  and  $P'_{2t}$  should equal the covariance between  $P_{1t}$  and  $P_{2t}$ . A comparison of  $\text{cov}(P_{1t}, P_{2t})$  with an estimated  $\text{cov}(P'_{1t}, P'_{2t})$ , which should (except for estimation error) be the same, is thus a valid way of testing the efficient markets model. The problem comes in interpreting violations of the efficient markets relation: we cannot take

$\text{cov}(P_{1t}, P_{2t})$  greater than  $\text{cov}(P'_{1t}, P'_{2t})$  as evidence of excess covariability. Suppose, for example, that prices are not set by  $P_t = E_t P_t^*$  but by  $P_t = E_t P_t^* + w_t$ , where  $w_t$  is a 'noise' vector whose variance matrix is diagonal (noise in one asset is independent of noise in the other asset) and which is independent of  $E_t P_t^*$  (noise is independent of true fundamentals). If  $P'_{it}$  ( $i = 1, 2$ ) is taken as the projection of  $P_{it}^*$  on  $P_{it}$ , then we will find that (by usual errors in variables results) the coefficient on price  $P_{it}$  is less than one so that the fitted value  $P'_{it}$  does not equal  $P_{it}$ . Thus, the efficient markets model is (correctly) found to be violated. However, it would be incorrect to infer that it is violated because of excessive covariance between  $P_{1t}$  and  $P_{2t}$ . The covariance between  $P'_{1t}$  and  $P'_{2t}$  will be less than the covariance of  $P_{1t}$  and  $P_{2t}$ , and yet clearly the covariance between  $P_{1t}$  and  $P_{2t}$  is quite right.

We want instead to put bounds on the covariance between  $P_{1t}$  and  $P_{2t}$  that are violated only when there is in fact excess covariance between the asset prices, and yet we still want to use information about  $\text{var}(P')$ . We can write  $P_t = P'_t + v_t$ , where the vector  $v_t$  is uncorrelated with  $P'_t$  since it represents an error unforecastable from the subset of information used to compute  $P'_t$ . To put an upper (lower) bound on  $\text{cov}(P_1, P_2)$  we must solve the nonlinear programming problem to maximize (minimize) it in terms of the three elements of  $\text{var}(P)$  subject to the inequality restrictions implicit in  $\text{var}(P) - \text{var}(P')$  and  $\text{var}(P^*) - \text{var}(P)$  both positive semidefinite, in other words, to maximize (minimize) in terms of the three elements of  $v$  such that  $\text{var}(v)$  and  $\text{var}(P^*) - \text{var}(v)$  are positive semidefinite, where  $\text{var}(P^*) \equiv \text{var}(P^*) - \text{var}(P')$ . This is really essentially the same maximization problem that we discussed in the preceding section, and the bounds implied by the solution to this problem and by the solution to the corresponding minimization problem are:

$$\begin{aligned} \text{cov}(P'_1, P'_2) + (\text{cov}(\varepsilon_1^*, \varepsilon_2^*) - \sigma(\varepsilon_1^*)\sigma(\varepsilon_2^*))/2 &\leq \text{cov}(P_1, P_2) \\ &\leq \text{cov}(P'_1, P'_2) + (\text{cov}(\varepsilon_1^*, \varepsilon_2^*) + \sigma(\varepsilon_1^*)\sigma(\varepsilon_2^*))/2 \end{aligned} \quad (4)$$

This inequality can put much tighter bounds on the warranted covariance between  $P_{1t}$  and  $P_{2t}$ . Suppose, for example,  $\text{var}(P^*)$  is the identity matrix, so that by (3)  $\text{cov}(P_1, P_2)$  can range between  $-0.5$  and  $+0.5$ . Suppose, however, that  $\text{var}(P')$  has all four of its elements equal to  $0.5$ . Then the upper bound to  $\text{cov}(P_1, P_2)$  is  $0.5$ , the lower bound is zero: the extra information reduced the range of warranted covariances by a half; moreover, in this case we know that the upper bound to the covariance between  $P_{1t}$  and  $P_{2t}$  is exactly equal to the covariance between  $P'_{1t}$  and  $P'_{2t}$ .

Knowing  $\text{var}(P')$  now enables us to put bounds on the correlation between  $P_{1t}$  and  $P_{2t}$ . Since  $v_t$  is uncorrelated with  $P'_t$  and we have:

$$\text{corr}(P_1, P_2) = \frac{\text{cov}(P'_1, P'_2) + \text{cov}(v_1, v_2)}{((\sigma(P'_1)^2 + \sigma(v_1)^2)(\sigma(P'_2)^2 + \sigma(v_2)^2))^{.5}} \quad (5)$$

We can put maximum and minimum values on this function with respect to

$\text{var}(v)$  subject to the restriction that  $\text{var}(v)$  and  $\text{var}(\varepsilon^*) - \text{var}(v)$  are both positive semidefinite.<sup>6</sup> This will give us bounds on the correlation between  $P_1$  and  $P_2$  that are analogous to the bounds (3) and (4) above. Plainly, so long as  $\text{cov}(P_1, P_2)$  is non-zero then this procedure will put some meaningful bounds on the correlation between  $P_1$  and  $P_2$ . Since  $P_t = P'_t + v_t$  where  $P'_t$  and  $v_t$  are uncorrelated, and since the variance matrix of  $v_t$  is limited by  $\text{var}(\varepsilon)$ , there is no way that perfect positive or perfect negative correlation between  $P_1$  and  $P_2$  can be achieved. By a similar argument, if the second asset is the market portfolio, we can place bounds on the beta between the two assets.

We will discuss below a present value model of stock prices that will allow us to compute  $\text{var}(P')$  and  $\text{var}(P^*)$  for a certain transformation of stock prices. We will then compute  $\text{cov}(P_1, P_2)$  and compare this with  $\text{cov}(P'_1, P'_2)$  as well as the bounds in (4) and compute  $\text{corr}(P_1, P_2)$  and compare this with  $\text{corr}(P'_1, P'_2)$  as well as the bounds implied by the maximization of (5).

#### 4. The data

For the US, the annual stock price is the Standard and Poor Composite Stock Price Index for January of the year. The dividend is total dividends per share adjusted to index, four quarter total, fourth quarter of the year, backdated before 1926 using the dividend series in Cowles (1939). The interest rate in the US is the continuously compounded annual return on 4–6 month prime commercial paper computed from January and July commercial paper rates assuming a six-month maturity. For the UK the annual stock price is the Barclay de Zoete Wedd (BZW) stock price index for the end of the preceding year, and the dividend is the associated BZW dividend series for the year. The UK interest rate is the three-month prime bank bill rate, averaged over the year, as a continuously compounded return. These are the same series as used in Shiller and Beltratti (1992).

For both the US and the UK we shall de-trend stock prices in each year by using as  $P_t$  and  $P_t^*$  the log of the price and present value respectively divided by the dividend for the preceding year. This differs from Shiller (1989), where prices were de-trended by dividing by a long moving average of dividends, and the resulting ratio was not logged. The dividing of nominal prices by nominal dividends serves to put the data in real terms: the variable  $P_t$  may be regarded also as the log of real price divided by real dividend, where the deflator used for both is the same.

#### 5. The present value relation

We will set up our present value relation, for the econometric work that follows, as a sort of dynamic Gordon model replacing the original Gordon model (1962), which was a steady-state growth path condition. Let us first review and recall the Gordon model. The Gordon model asserts that price

<sup>6</sup> This will be done by means of numerical methods described in Section 6.

is the present value of dividends with constant discount rate  $R$  and that dividends grow at a constant rate  $g$ . That is, the model assumes that the price  $P_0$  of a share at time 0 is given by:

$$P_0 = \int_0^{\infty} d_0 e^{gt} e^{-Rt} dt$$

Then, evaluating the integral, the dividend-price ratio  $d_0/P_0$  equals  $R - g$ . ( $R$  must be greater than  $g$  for the integral to converge.) By the Gordon model, then, high (low) dividend price ratios have either of two interpretations. Either the rate  $R$  used to discount future dividends is high (low), or the growth rate of dividends is low (high).

The Gordon model, unfortunately, was derived under assumptions that limit its usefulness, unless it is modified. The most natural application of the Gordon model is to interpret time variation in dividend-price ratios. One is tempted to interpret times of high dividend-price ratios as times when the discount rate  $R$  is high and/or the rate of growth of dividends  $g$  is low. But, the original Gordon model does not apply if the growth rate of dividends or the discount rate is not constant through time.

So, we shall use here a dynamic modification of the Gordon model, which allows time variation both in the discount rate and the rate of growth in dividends. The variant that we shall use will be in discrete time rather than continuous time, since our data are sampled at discrete intervals, and the variant will preserve essential linearity so that the model fits in well with linear time series methods. Moreover, it will be convenient to couch the analysis in terms of price-dividend ratios rather than dividend-price ratios.

We shall use a log-linearized version of the present-value model, developed by Campbell and Shiller (1988a, b). The model is:

$$P_{st} = E_t P_{st}^*$$

where

$$P_{st}^* \equiv \sum_{n=0}^{\infty} \rho_s^n G_{st+n} + k_s / (1 - \rho_s) \quad (6)$$

and where  $s = \text{UK (United Kingdom), US (United States)}$ . Here,  $P_{st}$  is the log price-dividend ratio for country  $s$ ,  $G_{st}$  is defined as  $\Delta d_{st} - i_{st}$ ,  $\Delta d_{st}$  is the change from the preceding period of log nominal dividends in country  $s$ ,  $i_{st}$  is the nominal one-period interest rate in country  $s$ , and  $k_s$  and  $\rho_s$  are constants of linearization (see Campbell and Shiller, 1988a). Note that  $G_{st}$  is (minus) the one-period counterpart to the  $R - g$  in the Gordon model: in effect the dynamic Gordon model makes the log price-dividend model an expectation of a moving average of future one-period  $R - g$  terms. For each country,  $\rho_s$  was taken to be  $\exp(\bar{g}_s - \bar{R}_s)$ , where  $\bar{g}_s$  is the average rate of growth of dividends and  $\bar{R}_s$  is the average return on stocks over the sample, and  $k_s$  does not affect our analysis when we calculate a time series for  $P_{st}^*$ . Equation (6) says that the log of the



price divided by dividend (January log price minus the log total dividends over the preceding year)  $P_{st}$  is equal to the expectation at time  $t$  of future *ex post* value  $P_{st}^*$ .

Equation (6) says simply that stock prices will be high relative to dividends when dividends are expected to grow more than average and/or short-term interest rates are expected to be low in the not-to-distant future, where not-to-distant is defined in terms of the discount parameter  $\rho_s$ . By this model the log price–dividend ratio will be stationary if the fundamentals are themselves stationary.

Note also that by writing equation (6) in the form  $P_{st} = E_t P_{st}^*$  we can also apply the analysis of the preceding sections. The covariances and correlations between price–dividend ratios ( $P_{st}$ ) are related to the covariances and correlations of the *ex post* values ( $P_{st}^*$ ). The price–dividend ratios in two different countries cannot co-vary or correlate highly unless the *ex post* values do, in accordance with the above inequalities. This means then that the price–dividend ratios in two countries cannot co-vary highly with each other unless the discount rates and/or growth rates in the not-too-distant future co-vary highly across the two countries. Our analysis below will inquire whether the cross-country covariance or correlation of price–dividend ratios can be reconciled in this way with the actual covariances or correlations across countries in these variables, in accordance with dynamic Gordon model (6).

To achieve this end, we will have to set up some vector-autoregressive econometric methodology. A problem in testing this model is that the summation in (6) extends to infinity, and hence all the terms in the summation are not observed; our econometric methodology must address this problem.

## 6. Econometric methodology

The bounds on the covariances and correlations which we can derive are based on the moments of the vector of *ex post* values. We will use two different methods to compute these moments. The first one is the same method usually followed in the literature and proposed by Shiller (1989); it is based on calculating a time series for  $P_{st}^*$  subject to a terminal condition which says that  $P_{sT}^*$  in the last year  $T$  of the sample is equal to the actual  $P_{sT}$  on that date:

$$P_{st}^* = \sum_{k=0}^{T-k-1} \rho_s^k G_{st+k} + \rho_s^{T-t} P_{sT}, \quad s = \text{US, UK} \quad (7)$$

From these time series one can then estimate the sample covariance matrix for  $P_t^* = [P_{\text{US}t}^*, P_{\text{UK}t}^*]'$  to use in the inequality expressions.

The second method which we use does not involve computation of a time series for  $P_t^*$ . Defining  $G_t = [G_{\text{US}t}, G_{\text{UK}t}]'$ , and defining  $\rho$  as a  $2 \times 2$  matrix with  $\rho_{\text{US}}$  and  $\rho_{\text{UK}}$  on the diagonal, then from (7)  $P_t^* \Sigma(k=0, \infty) \rho^k G_{t+k}$ , (plus a

constant which we will disregard) and so  $\text{var}(P_t^*)$  is given by:

$$\text{var}(P_t^*) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \rho^j \text{cov}(G_{t+j}, G'_{t+k}) \rho^k \quad (8)$$

Using (8) to estimate  $\text{var}(P_t^*)$  of course involves estimating the complete autocovariance function for  $G_t$  at all leads and lags. Unfortunately, for a given sample of data, we cannot estimate all the infinite series of covariance matrices, and we have to truncate the estimated autocovariance function after a finite number of lags. We report results for lag  $n/4$  as well as for lag 30. Note that in (8), future covariances are multiplied by the terms  $\rho_{\text{US}}$  and  $\rho_{\text{UK}}$ , which are less than one; this means that autocovariances at long lags are already given less weight because of the very definition of the vector  $P_t^*$ , so that truncation is not likely to affect the results much.

As to the bound on the covariance and the correlation which we have derived for the case when some forecasting information is available for statistical analysis, that is (4), one can see that the information contained in the perfect foresight price must be supplemented with information contained in the statistical estimate of the fundamental price of the assets. An econometric model is therefore necessary to this purpose.

Following previous work by Campbell and Shiller (1988a, b) we use vector autoregressions to test the models and to calculate the expectations of future fundamentals given an *a priori* specified information set. In the case of a VAR of order 1<sup>7</sup> we consider the following vector:

$$x_t = [P_{\text{US}t}, G_{\text{US}t-1}, P_{\text{UK}t}, G_{\text{UK}t-1}]' \quad (9)$$

where variables are demeaned. By selecting this vector for the vector autoregressive model, we have chosen as an information set the price-dividend ratios and the variables  $G$ , which are the one-period growth rates of dividends minus the one-period interest rates. Note that the  $G_{st}$ ,  $s = \text{US}, \text{UK}$ , are lagged in this vector, so that it contains only information known by the agents at the beginning of period  $t$ .

We assume an autoregressive form for the vector  $x$ :

$$x_{t+1} = Ax_t + a_t \quad (10)$$

where  $a_t$  is a white noise term with a covariance matrix which can have non-zero contemporaneous correlations. Now, the model (10) implies that the optimal forecast at time  $t$  of  $x_{t+1}$  is  $Ax_t$ , and the optimal forecast of  $x_{t+k}$  is  $A^k x_t$ . It follows that the optimal forecast of the present value  $x_t + \rho x_{t+1} + \rho^2 x_{t+2} + \dots$  is  $(I - \rho A)^{-1} x_t$ . We can pick out of this vector of optimal forecasts of present values the  $i$ th element by pre-multiplying the optimal

<sup>7</sup> We consider in the text only the first-order VAR case, since higher-order VARs can be easily treated with the same methodology after putting them into a first order 'companion form' VAR, as described in Campbell and Shiller (1988a, b).

forecast vector by  $ei'$  where  $ei$  is a vector of zeros apart from the  $i$ th element which is equal to 1.

The model (10) therefore implies:

$$P_{st} = P'_{st} \quad s = \text{US, UK}$$

where

$$P'_{US,t} \equiv e2'A(I - \rho_{US}A)^{-1}x_t \quad (11a)$$

$$P'_{UK,t} \equiv e4'A(I - \rho_{UK}A)^{-1}x_t \quad (11b)$$

Here, the prime after the  $P$  is used as it was in Section 3 above, to denote the expectation of the present value given only the information  $x_t$ , i.e., the price-dividend ratio implied by the dynamic Gordon model (6) and the time series model (10). The  $P'$  here is the same as in Campbell and Shiller (1988a, b), where  $P'$  was referred to as the 'theoretical' price-dividend ratio; it may also be called the warranted price-dividend ratio. Now, since  $P_{st}$  and  $P_{kt}$  are elements of  $x_t$ , they can be written in terms of  $x_t$  as  $e1'x_t$  and  $e3'x_t$  respectively. Substituting into the left-hand sides of (11a) and (11b) we can then cancel  $x_t$  from the equations and the regressions (11a, b) thus imply the following cross-equation restrictions on the estimated matrix  $A$ :

$$e1'(I - \rho_{US}A) = e2'A \quad (12a)$$

$$e3'(I - \rho_{UK}A) = e4'A \quad (12b)$$

We test these linear restrictions by means of Wald tests. Beyond testing the models, if we are willing to identify expectations with linear projections, we can use (11a, b) to derive the expectations of future fundamentals under the hypothesis that the model is true (see Campbell and Shiller 1988a, b), and then use these estimated values to compute what in the previous sections was defined with the variable  $P'$ .

In order to consider the possibility of small sample bias we calculate empirical distributions for all the statistics that we report in the tables<sup>8</sup> by a Monte Carlo experiment which generates 2,000 series of the variables in the vector  $x$  subject to the restrictions that the models for the two assets are true. We report both numerical standard errors for the statistics, and the  $p$ -value corresponding to the empirical distribution.

We can also use our  $P'$  to calculate the bounds in expressions (4). Again we can use two methods to compute the covariance bounds. One possibility is to compute  $P^*$  with a terminal value, compute  $P'$  from the VAR, calculate  $\varepsilon^* = P^* - P'$ , as with expression (7). From these time series one can compute the variance matrices  $\text{var}(P^*)$ ,  $\text{var}(P')$ , and  $\text{var}(\varepsilon^*)$ . This guarantees that all matrices are positive semidefinite. The second possibility is to calculate  $\text{var}(P^*)$  from the covariance matrix of  $G_t$  using (8), and then calculate  $\text{var}(\varepsilon^*)$  as the

<sup>8</sup> These statistics are the Wald tests, and the difference between theoretical and actual covariances and correlations.

difference between  $\text{var}(P^*)$  and  $\text{var}(P')$ , where the last is computed from the time series of  $P'$ .

As to the correlations between the two assets, we generate upper and lower bounds by means of numerical methods. We use a Monte Carlo program that generates random positive definite matrices  $\text{var}(v)$ , and which tests then if  $\text{var}(\varepsilon^*) - \text{var}(v)$  is also positive semidefinite. If it passes the test, the program calculates the correlation coefficient using expression (5). After repeating the exercise 4,000 times we pick the highest and the lowest correlation. In particular, we make the diagonal elements of  $\text{var}(v)$  uniform from zero to corresponding diagonal elements of  $\text{var}(\varepsilon^*)$ . In each iteration we compute from these diagonal elements  $\sigma(v_1)\sigma(v_2)$ , and make off diagonal elements of  $\text{var}(v)$  uniform from  $-\sigma(v_1)\sigma(v_2)$  to  $+\sigma(v_1)\sigma(v_2)$ . So  $\text{var}(v)$  is positive semidefinite, and the diagonal elements of  $\text{var}(\varepsilon^*) - \text{var}(v)$  are non-negative. We then only need to check that the determinant of  $\text{var}(\varepsilon^*) - \text{var}(v)$  is non-negative in each iteration.

Both for the covariance bounds and the correlation bound we calculate standard errors by means of Monte Carlo simulations. In this case we generate 4,000 series of variables from the estimated VAR and we use them to calculate the standard errors of the bounds across iterations.

## 7. Results

Table 1, panel A, shows that the Wald tests usually reject the restrictions (12), and this is similar to previous results (Campbell and Shiller 1988, Beltratti 1989, and Shiller and Beltratti 1992). We report both asymptotic  $p$ -values, and  $p$ -values from the empirical distribution function obtained from the restricted model. Note that the asymptotic standard errors sometimes over-reject the model, but the differences are minimal even for large order VARs. We can thus reject the model represented by equations (6) and (10), but it remains to be seen whether the rejection might have something to do with excess co-movement between countries, and for this we turn to the inequalities.

Table 1 Panel B shows that the correlation between the estimated warranted prices  $P'$  tends to be higher than the correlation between prices, but that the covariance between the estimated warranted prices tends to be lower than the covariance between the actual prices. This sort of difference between the results with covariances and with correlations has been noted before (see for example, Campbell and Shiller, 1988b); the difference reflects the estimated 'excess volatility' of both markets, which drives up covariances but not correlations of actual prices relative to warranted values. Of course, these results are in no way tests of the efficient markets model; essentially they make no account of the possibility that the market may have superior information from that used to make estimated  $P'$ , and hence we turn to the covariance and correlation bounds.

Table 2 reports results for the covariance bound that can be derived when no forecasting information is available for statistical analysis, that is the bounds given in expression (3). The actual covariance is within the bounds

TABLE 1  
Wald tests and co-movement measures

<i>Panel A: results from VAR estimation</i>			
<i>Tests of restrictions expressions (10)</i>			
Lags	1	2	3
Country: US			
Wald test:			
asymptotic $p$ -value	0.009	0.014	0.015
$p$ -value from e.d.f.	0.014	0.026	0.029
Country: UK			
Wald test:			
asymptotic $p$ -value	0.000	0.000	0.000
$p$ -value from e.d.f.	0.000	0.000	0.000
<i>Panel B: Co-movements between stock markets</i>			
<i>Corr(<math>P_{US}, P_{UK}</math>): 0.470 Cov(<math>P_{US}, P_{UK}</math>): 0.033877</i>			
<i>Warranted co-movements estimated using expression (11):</i>			
Corr( $P'_{US}, P'_{UK}$ ) – Corr( $P_{US}, P_{UK}$ )	0.113	0.393	0.417
numerical std. error	0.339	0.205	0.179
std. error from e.d.f.	0.184	0.224	0.241
Cov( $P'_{US}, P'_{UK}$ ) – Cov( $P_{US}, P_{UK}$ )	–0.029	–0.018	–0.011
numerical std. error	0.004	0.013	0.016
std. error from e.d.f.	0.023	0.027	0.028

Note: Sample period is 1919–89.

TABLE 2  
Covariance bounds from equation (3) in the text

	<i>Actual Cov(<math>P_{US}, P_{UK}</math>) = 0.034</i>	
	<i>Lower bound</i>	<i>Upper bound</i>
(i) Var( $P^*$ ) computed from time series of $P^*$ , using expression (7).	–0.008 (0.012)	0.045 (0.027)
(ii) var( $P^*$ ) is computed from the estimated autocovariance function of fundamentals up to 30 lags, using expression (8).	–0.010 (0.012)	0.036 (0.019)
(iii) Var( $P^*$ ) is computed from the estimated autocovariance function of fundamentals up to $(n/4)$ lags, where $n$ is the number of observations, using expression (8):	–0.005 (0.012)	0.048 (0.027)

Note: The numbers in parentheses are standard errors obtained from a Monte Carlo simulation. Sample period is 1919–89.

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TABLE 3  
Covariance bounds from equation (4) in the text

	<i>Actual Cov</i> ( $P_{US}, P_{UK}$ ) = 0.034	
	<i>Lower bound</i>	<i>Upper bound</i>
(i) $\text{Var}(P^*)$ computed from time series of $P^*$ , using expression (7).		
<i>Order of the VAR</i>		
1	-0.004 (0.012)	0.049 (0.025)
2	0.003 (0.015)	0.046 (0.030)
3	0.005 (0.027)	0.050 (0.042)
(ii) $\text{Var}(P^*)$ is computed from the estimated autocovariance function of fundamentals up to 30 lags, using expression (8).		
1	-0.003 (0.013)	0.034 (0.019)
2	0.003 (0.014)	0.034 (0.018)
3	0.008 (0.018)	0.033 (0.020)
(iii) $\text{Var}(P^*)$ is computed from the estimated autocovariance function of fundamentals up to $(n/4)$ lags, where $n$ is the number of observations, using expression (8).		
1	0.001 (0.012)	0.047 (0.025)
2	0.008 (0.014)	0.047 (0.018)
3	0.013 (0.017)	0.046 (0.019)

Note: Each sub-panel reports results from VAR of order 1, 2 and 3. The numbers in parentheses are standard errors obtained from a Monte Carlo simulation. Sample period is 1919–89.

in all cases, but close to the upper bound. There is not much difference between the results obtained by estimating the covariance matrix of the time series of  $P^*$  or by estimating the autocovariance function of fundamentals when only  $(n/4)$  terms are included in the last. However, when 30 terms are considered the upper bound gets closer to the actual value.

The same structure of results appears in Table 3, when the information set contained in the estimated VAR is used for the covariance bounds given by expression (4). Again, the actual covariance is usually within the bounds. Again a long estimated autocovariance function tends to lower the upper bound. Results from VARs of order 1, 2 and 3 are not very different from each other.

Finally, also the actual correlations, shown in Table 4 are in general inside the bounds computed, using expression (5), by Monte Carlo methods. When only one lag is used in the vector autoregression, the estimated correlation

TABLE 4  
Correlation bounds from equation (5) in the text

<i>Actual corr</i> ( $P_{US}, P_{UK}$ ) = 0.470		
	<i>Lower bound</i>	<i>Upper bound</i>
(i) $\text{Var}(P^*)$ computed from time series of $P^*$ , using expression (7).		
<i>Order of the VAR</i>		
1	-0.212 (0.268)	0.929 (0.106)
2	0.167 (0.298)	0.933 (0.094)
3	0.187 (0.274)	0.932 (0.058)
(ii) $\text{Var}(P^*)$ is computed from the estimated autocovariance function of fundamentals up to 30 lags, using expression (8).		
1	0.135 (0.185)	0.929 (0.098)
2	0.437 (0.213)	0.940 (0.082)
3	0.415 (0.256)	0.919 (0.061)
(iii) $\text{Var}(P^*)$ is computed from the estimated autocovariance function of fundamentals up to $(n/4)$ lags, where $n$ is the number of observations, using expression (8).		
1	-0.148 (0.268)	0.899 (0.106)
2	0.183 (0.279)	0.932 (0.093)
3	0.324 (0.248)	0.896 (0.059)

Note: Each sub-panel reports results from VAR of order 1, 2 and 3. The numbers in parentheses are standard errors obtained from a Monte Carlo simulation. Sample period is 1919–89.

bounds are extremely wide, allowing almost anything from no correlation to perfect positive correlation. The bounds are substantially tighter when more lags are introduced, reflecting the information available in the further lagged values.

## 8. Conclusions

We are unable to reject the hypothesis that the covariance and correlation between the US and UK log price-dividend ratios is in accordance with the present value model. The bounds on covariances and correlations are quite wide and usually embrace the actual covariance and correlations. Despite the tendency of the US and UK stock markets to move somewhat together, even to undergo boom markets and stock market crashes together, and despite the casually asserted opinion that there could be no objective fundamental reason

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for these comovements, we do not find any evidence of excessive comovements. The problem is that informative pooling could justify a large amount of comovement between the markets: information pooling could justify the covariance we documented between the US and UK markets even though there is less covariance between fundamentals in the two markets.

This does not mean that if a larger information set were used we might have been able to get narrower bounds on covariances and correlations, and might then have been able to reject the model. Note also that in this paper, in contrast with some results in Shiller (1989), time varying interest rates are used to discount in the present value formulae.

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## REFERENCES

- BELTRATTI, A. E. (1989). 'Essays in Stock Market Efficiency and Time-Varying Risk Premia', unpublished Ph.D. dissertation, Yale University.
- CAMPBELL, J. Y. and SHILLER, R. J. (1988a). 'The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors', *Review of Financial Studies*, 1, 195–228.
- CAMPBELL, J. Y. and SHILLER, R. J. (1988b). 'Stock Prices, Earnings, and Expected Dividends', *Journal of Finance*, 43, 661–76.
- CHEN, N. F., ROLL, R. and ROSS, S. A. (1986). 'Economic Forces and the Stock Market', *Journal of Business*, 59, 383–403.
- COWLES, A. (1939). *Common Stock Indexes*, 2nd ed., Principia Press, Bloomington.
- FAMA, E. F. and FRENCH, K. R. (1989). 'Business Conditions and Expected Returns on Stocks and Bonds', *Journal of Financial Economics*, 25, 23–49.
- FRIEDMAN, M. and SCHWARTZ, A. J. (1982). *Monetary Trends in the United States and the United Kingdom*, The University of Chicago Press, Chicago.
- GORDON, M. J. (1962). *The Investment, Financing, and Valuation of the Corporation*, Irwin, Homewood, IL.
- KING, M. A. and WADHWANI, S. (1990). 'Transmission of Volatility between Stock Markets', *The Review of Financial Studies*, 3, 5–33.
- KING, M. A., SENTANA, E. and WADHWANI, S. (1990). 'Volatility and Links between National Stock Markets', National Bureau of Economic Research Working Paper No. 3357, Cambridge MA.
- PINDYCK, R. S. and ROTEMBERG, J. J. (1990). 'Do Stock Prices Move Together Too much?', National Bureau of Economic Research Working Paper No. 3324, Cambridge MA.
- PINDYCK, R. S. and ROTEMBERG, J. J. (1988). 'The Excess Co-Movement of Commodity Prices', National Bureau of Economic Research Working Paper No. 2671, Cambridge MA.
- SHILLER, R. J. (1989a). 'Comovements in Stock Prices and Comovements in Dividends', *Journal of Finance*, 44, 719–29.
- SHILLER, R. J. (1989b). *Market Volatility*, MIT Press, Cambridge MA.
- SHILLER, R. J. and BELTRATTI, A. E. (1992). 'Stock Prices and Bond Yields: Can Their Comovements be Explained in Terms of Present Value Models?', *Journal of Monetary Economics*, 30, 25–46.