Differentiated Products

While homogenous goods is a convenient assumption for many models, it is frequently violated in practice. Many markets feature goods that are not identical, they vary in quality, features, reliability, reputation and/or geographic location. Markets of literally identical goods seem to be relatively rare, especially once differences in seller’s locations and reputations are taken into account.
• Demand analysis obviously becomes much more complicated, as we can no longer think of a single market demand curve, but rather a demand system
• Even more plausible to think of the market as an oligopoly,
• How it is that products come to be differentiated?
Review of Theory.

Here we discuss various approaches to demand, and then discuss the static oligopoly equilibrium.

There are two traditions in deriving demand. The first tradition considers a representative consumer who has a taste for consuming a variety of products. Typically, the demand curve is derived from a well-specified utility function that features decreasing marginal utility from the consumption of each good in the market. This provides the incentive for the representative consumer to spread consumption across a variety of goods. The reason that goods are differentiated is typically buried in the parameters of the utility function, rather than being made explicit.
A second tradition locates goods in a space of product characteristics. Consumers have heterogeneous tastes and place differing utility weights on the different product characteristics. Each consumer is often modeled as buying at most one unit of the good, yielding a “discrete choice” model of demand at the consumer level. Product variety is then a response to the variety of consumers preferences, rather the the “taste for variety” of a representative consumer. Aggregate (market level) demand is then found by summing up the demands of the individual consumers.
One advantage of the characteristics approach is that it is easy to incorporate information about household-level demand. The representative consumer approach would face the problem of explaining why different households make different choices. However, there is no reason why the households level data must be modelled according to a discrete choice framework, as indicated by the discussion of Jorgensen style translog demand functions in an earlier section.

The choice between discrete and continuous models of demand will typically be decided by the nature of the market at hand.
Representative Consumer Models.

Here we discuss Dixit-Stiglitz style models (see Tirole).

[On the board]
Discrete Choice Models.

Many of the product differentiation models we consider are of the discrete-choice form. An advantage of these models is that they build demand from a well-specified utility for the characteristics of products. An unfortunate restriction is that they usually constrain each consumer to consider buy at most one unit of a good. This restriction can be relaxed: e.g. Hendel’s 1994 Harvard Ph.D. thesis. Examples of discrete choice models used in IO theory include the Hotelling and vertical models. Anderson, DePalma and Thisse (1992) provide a lengthy discussion of discrete choice models as used in the theory of product differentiation.
A general discrete choice model starts with by specifying the utility of consumer i for product j as

$$u_{ij} = U(x_j, p_j, \nu_i),$$

where $x_j$ is a vector of product characteristics, $p_j$ is the price of the product and $\nu_i$ is a vector of consumer characteristics. (Rather than model utility directly as a function of price, it might be preferable to model it as a function of expenditures on other products and then derive the “indirect” utility $U$ as a function of price.)
As example, the utility function in the Hotelling model with quadratic transportation costs is

\[ u_{ij} = \bar{u} - p_j - (x_j - \nu_i)^2, \]

where \( x_j \) is the location of the product along the line and \( \nu_i \) is the location of the consumer. [brief derivations of Hotelling demand function here: note that demand is the integral over a region of the \( \nu \)'s.]
Hotelling (on the board)
In the vertical model (Shaked and Sutton 1982, Bresnahan 1987),

\[ u_{ij} = \nu_i x_j - p_j \]

where \( x_j \) is “quality”, and \( \nu_i \) is the consumers “taste” for quality.  
[Brief derivation of demand on the board. Note that as in Hotelling, a small variation in price or quality affects only the firms two “neighbors”. This is a result of the one-dimensional product space.]
Or, the model can be written as

\[ u_{ij} = \delta_j - \alpha_i p_j, \]

We normalize the utility of the outside alternative \((u_{i,0})\) to zero. Order the goods in terms of increasing price. Then good \(j\) is purchased \(\text{iff}\ \ u_{ij} > u_{ik}, \ \forall k \neq j.\), or equivalently

\[ \delta_j - \alpha_i p_j > \delta_k - \alpha_i p_k, \ \Rightarrow \ \alpha_i (p_j - p_k) < \delta_j - \delta_k, \ \forall k \neq j. \]
Recall that \((p_j - p_k)\) is positive if \(j > k\) and negative otherwise. So a consumer endowed with \(\alpha_i\) will buy product \(j\) iff

\[
\alpha_i < \min_{k<j} \frac{\delta_j - \delta_k}{(p_j - p_k)} \equiv \Delta_j(\delta, p), \quad \text{and}
\]

\[
\alpha_i > \max_{k>j} \frac{\delta_k - \delta_j}{(p_k - p_j)} \equiv \Delta_j(\delta, p). \quad (1)
\]
These formula assume that $0 < j < J$. However if we set

$$\Delta_0 = \infty, \text{ and } \Delta_J = 0.$$ 

they extend to the $j = 0$ (the outside good) and $j = J$ cases.
If the cdf of $\alpha$ is $F(\cdot)$, then the market share of product $j$ is

$$s_j(x, p, \xi) = (F(\Delta_j(x, p, \xi)) - F(\Delta_j(x, p, \xi)) 1[\Delta_j > \Delta_j],$$

where $1[\cdot]$ is the indicator function for the condition in the brackets.
Note that if $\overline{\Delta} \leq \Delta$, then $s_j(\cdot) = 0$. Since the data has positive market shares, the model should predict positive market shares at the true value of the parameters. However the pure characteristics model behaves unlike the standard models in that it will predict zero market shares for some parameter values (e.g. any parameter vector which generates an ordering which leaves one product with a higher price but lower quality than some other product will do).
Vertical-model-like utility functions can be extended to multiple characteristics. For example, an analog of the Hotelling model, with $K$ characteristics, is

$$u_{ij} = \bar{u} - p_j - \sum_{k=1}^{K} \alpha_k (x_{jk} - \nu_{ik})^2,$$

while an extension of the vertical model is the “pure random coefficients model”:

$$u_{ij} = \left[ \sum_{k=1}^{K} \nu_{ik} x_{jk} \right] - p_j.$$
A traditional econometric specification is:

\[ u_{ij} = x_j \beta_i - \alpha_i p_j + \epsilon_{ij} \]

where the consumer “tastes” are given by

\[ \nu_i = (\beta_i, \alpha_i, \epsilon_{i1}, \epsilon_{i2}, \ldots, \epsilon_{iJ}) \]

In the traditional econometric specification, the \( \epsilon_{ij} \) are assumed to be i.i.d. across products and consumers. In the simplest case, there are no random coefficients on the product characteristics, \( (\beta_i = \beta \text{ and } \alpha_i = \alpha) \) and the \( \epsilon_{ij} \) have the “type 2 extreme value” distribution

\[ F(\epsilon) = e^{-e^{-\epsilon}}. \]
This gives the traditional logit model, where the probability that good \( j \) is purchased (i.e. the market share of product \( j \)) is

\[
    s_j = \frac{e^{\delta_j}}{\sum_{r=1}^{J} e^{\delta_r}},
\]

with \( \delta_j = x_j \beta - \alpha p_j \). If we add an “outside good” with \( \delta_j = 0 \), the logit market share becomes:

\[
    s_j = \frac{e^{\delta_j}}{1 + \sum_{r=1}^{J} e^{\delta_r}}.
\]
While the logit market share is easy to calculate, the model has unintuitive properties. In particular, cross-price effects do not depend on the degree to which the products have similar $x_j$’s, but only on the values of the sum $\delta_j$. 
The counter-intuitive substitution patterns do not come only from the specific distributional assumption in the logit model, but from the assumption that the only variance in consumer tastes comes through the i.i.d. product-specific terms $\epsilon_{ij}$. Since these terms are i.i.d., there is no source of correlation in consumer tastes across similar products.
When variance is added to the term $\beta_i$ and/or $\alpha_i$, then substitution patterns can become more reasonable. Now, a consumer who buys a good with a large value of some characteristic is more likely than the average consumer to have as a second choice another good with a large value of that characteristic.
However, as long as the distribution of $\epsilon$ is i.i.d., there are effectively as many product characteristics as there are products. Indeed, we can think of $\epsilon_{ij}$ as the consumer taste for a product characteristic that is defined to be equal to one for product $j$ and zero otherwise. If the $\epsilon$’s have an unbounded distribution, then all goods are strict substitutes for one another (e.g. $\partial s_j / \partial p_r$ is positive for all $j$ and $r$.) Contrast this to the one-dimensional taste models (Hotelling, vertical) in which each product is a substitute with only 2 other “nearby” products.
Equilibrium

Having discussed some methods of deriving demand, we can turn to a consideration of equilibrium. The simplest models of product differentiation would consider a set of single product firms each producing a differentiated product. We could begin by specifying a demand system for this set of related products, together with cost functions and an equilibrium notion.
The usual assumption is Nash in prices. To analyze the case of equilibrium with differentiable demand, note that the profits of firm \( j \) are given by
\[
\pi_j(p) = p_j q_j(p) - C_j(q_j(p)).
\]
The first-order condition is:
\[
q_j + (p_j - mc_j) \frac{\partial q_j}{\partial p_j} = 0.
\]
Note that we can rewrite this as
\[
p_j = mc_j + b_j(p),
\]
where the markup is
\[
b_j(p) = \frac{q_j}{\left| \frac{\partial q_j}{\partial p_j} \right|}
\]
We can write the product Lerner index in terms of the usual “inverse elasticity” rule.

\[
\frac{(p_j - mc_j)}{p_j} = \frac{1}{\eta_j},
\]

where \( \eta_j \) is the absolute value of the product-specific elasticity.
The second order-condition is given by:

\[
\frac{\partial q_j}{\partial p_j} + (p_j - m_{c_j}) \frac{\partial^2 q_j}{\partial p_j^2} - \frac{\partial m_{c_j}}{\partial q_j} \left( \frac{\partial q_j}{\partial p_j} \right)^2 < 0
\]

This is surely negative when demand slopes down and is concave and marginal cost does not slope down. If demand is too convex, we sometimes have trouble. If the s.o.c. holds everywhere, we have a sufficient (but not necessary) condition for a well-defined reaction function.
Differentiating the first-order condition, we find that prices are strategic complements when

\[ \frac{\partial q_j}{\partial p_k} + (p_j - mc_j) \frac{\partial^2 q_j}{\partial p_j \partial p_k} - \frac{\partial mc_j}{\partial q_j} \frac{\partial q_j}{\partial q_j} \frac{\partial q_j}{\partial p_k} \frac{\partial q_j}{\partial p_j} > 0 \]

This obviously holds in the linear demand case when the goods are substitutes, but need not hold in a non-linear case. This accounts for the general intuition that prices are strategic complements in the price-setting model.
Non-differentiable Demand

Some care must be taken in assuming differentiable demand functions, because many natural specifications result in discontinuous demand functions: for example our linear Hotelling model.
In a 1991 paper, Caplin Nalebuff (1991) consider a broad class of discrete-choice models and provide a result on the existence of equilibrium in such models. They also provide a partial characterization of the equilibrium. They require two assumptions on the distribution of consumer utility. The first requires utility to be linear in consumer characteristics; note that all of our examples above satisfy this requires (although the quadratic transport cost example requires some re-writing to see this.) The second assumption places a restriction on the density of consumer tastes which restricts the density to a set that is broader than the set of log-concave densities.
Quantity vs. Price Setting

In differentiated products models, one could also consider quantity setting firms, as in the Cournot model. In this case, the first-order condition becomes

\[ p_j + q_j \frac{\partial p_j}{\partial q_j} - m c_j = 0. \]

Note that \( \frac{\partial p_j}{\partial q_j} \) is the \( j^{th} \) diagonal element of the J by J matrix

\[
\left[ \frac{\partial q}{\partial p'} \right]^{-1},
\]

which is not the same as the inverse of \( \frac{\partial q_j}{\partial p_j} \) (except in the monopoly case). For many examples, the quantity-setting markup will be higher than the price setting markup.
The question of price vs. quantity setting in differentiated product models is asked, in a Kreps-Scheinkman like way, in J. Friedman (RAND, 1988) and I. Hendel (199?).