Question: How does the savings rate affect long-run growth?

We will answer this question using a very simple aggregate (or economywide) model of economic growth.

This model is based on the Nobel Prize-winning work of Robert Solow, an economist at MIT.
Remember the aggregate production function:

\[ Y = A K^\lambda L^{1-\lambda} \]

To keep things really simple (for now), let's set \( A = 1 \) and fix \( L = 1 \).

What this means in practice is that (at least for now), we are not going to consider changes in technology (\( A \)) or employment (\( L \)).
But we are going to consider changes in $K$ (capital formation), and these changes will cause $Y$ to change (because $Y = K^\lambda$).

Let’s study the shape of the aggregate production function (again, holding technology and employment constant).
This production function exhibits diminishing returns to capital: the extra output from a little bit more capital decreases as \( K \) increases.
Put differently, diminishing returns to capital means that the slope of the production function decreases as capital increases:

The slope of the tangent line is higher at $K_1$ than at $K_2$. 
The slope of the production function is called the marginal product of capital.

The marginal product of capital is the amount by which output increases when capital increases by a (very) small amount.

The declining marginal product of capital suggests that it will be difficult to generate sustained growth simply by increasing capital over time.
Sergey Brin, co-founder of Google, on diminishing returns to capital

Mr. Brin said that he saw no end to other innovations. “You might imagine the lower-hanging fruit has been picked,” he said, “but at the same time we have built ladders and are reaching for larger, higher-hanging fruit.”

From: New York Times Business Section, 10/20/06

Link to full article: http://www.nytimes.com/2006/10/20/technology/20google.html
The Solow Growth Model

1. At the beginning of every time period $t$ (a time period might correspond to 1 year), the economy has a stock of capital $K_t$.

2. The economy produces output $Y_t$ according to the aggregate production function: $Y_t = K_t^\alpha$.

3. Some of this output is consumed today and some of it is invested in future capital (investment here means physical capital formation).
Let's assume further (for the sake of simplicity), that the entire current stock of capital is used up during the course of production.

This means that if the economy does not invest today, there will be no capital with which to produce tomorrow.
Some Additional Notation

Let $c_t$ be (aggregate) consumption in period $t$.

Let $I_t$ be (aggregate) investment in period $t$.

All output in period $t$ is either consumed or invested:

$$Y_t = c_t + I_t$$
Usually, \( Y_t = C_t + I_t + G_t + NX_t \)

\[ \text{government spending} \quad \text{net exports} \]

But in this very simple model, we are ignoring government spending and we are imagining that the economy is closed (so that it does not trade with the rest of the world).
Key decision facing any economy: how to split today’s output between today (consumption) and tomorrow (investment).

Let’s assume that \( C_t = (1-s) Y_t \) where \( s \) is the economy’s savings rate.

Since \( C_t + I_t = Y_t \) (by definition), \( I_t = s \ Y_t \).
• In other words, aggregate savings, denoted by \( S_t \), equals \( s y_t \).

• In a closed economy, \( S_t = I_t \), so \( I_t = s y_t \).

• Because capital depreciates completely during production, investment \( (I_t) \) is the only source of capital goods in the future: \( K_{t+1} = I_t \). (Note: it takes one period to install new capital goods.)
Let's put all the equations together now.

\[
\begin{align*}
Y_t &= K_t \\
S_t &= S \cdot Y_t \\
I_t &= S_t \\
K_{t+1} &= I_t
\end{align*}
\]

\[
\begin{align*}
Y_{t+1} &= K_{t+1} \\
S_{t+1} &= S \cdot Y_{t+1} \\
I_{t+1} &= S_{t+1} \\
K_{t+2} &= I_{t+1}
\end{align*}
\]

\{ \text{period } t \}

\{ \text{period } t+1 \}
We can simplify the analysis by collapsing the equations for period $t$ into a single equation involving only the stock of capital:

$$K_{t+1} = S K_t$$

This is the law of motion for the behavior of the capital stock.
Another way to write the law of motion:

\[ K_{t+1} - K_t = sK_t^\alpha - K_t \]

change in the capital stock (\( \Delta K_{t+1} \))

So: \( \Delta K_{t+1} \) is positive if \( sK_t^\alpha > K_t \)

\( \Delta K_{t+1} \) is negative if \( sK_t^\alpha < K_t \)

\( \Delta K_{t+1} \) is zero if \( sK_t^\alpha = K_t \)
A Useful Graph

\[ y = k^\alpha \]

\[ I = s k^\alpha \]

When \( K = \bar{K} \),

\[ s k^\alpha = K, \text{ so} \]

\[ \Delta K = 0. \]

\( \bar{K} \) is the steady state value of capital.
Let the initial period be $t = 0$.

**If $K_0 = \bar{K}$, then the economy's capital stock remains at $\bar{K}$.

Period 0: $K_1 = sK_0^\alpha = s\bar{K}^\alpha = \bar{K}$

This equation defines $\bar{K}$

Period 1: $K_2 = sK_1^\alpha = s\bar{K}^\alpha = \bar{K}$

Period 2: $K_3 = sK_2^\alpha = s\bar{K}^\alpha = \bar{K}$

Periods 3, 4, 5, ... : more of the same
Solving for the steady state

\[ \bar{K} = s \bar{K}^\alpha \implies \bar{K} = s^{\frac{1}{1-\alpha}}. \]

The steady state depends on the savings rate \( s \) and the exponent \( \alpha \) in the aggregate production function.

The higher is \( s \), the higher is \( \bar{K} \).
What about consumption in the steady state?

Steady-state output $\bar{Y} = \bar{K}^\alpha$

Steady-state consumption $\bar{C} = (1-s) \bar{Y}$

$\quad = (1-s) \bar{K}^\alpha$

$\quad = (1-s) \bar{s}^{\frac{\alpha}{1-\alpha}}$

Is there a BEST CHOICE for the savings rate $s$?
THE GOLDEN RULE
(discovered by Edmund Phelps, this year's winner of the Nobel Prize in Economics)

At $s = 0$, $\bar{k} = 0$, so $\bar{c} = 0$
At $s = 1$, $\bar{c} = (1-s)\bar{Y} = 0$
At $s = \alpha$, $\bar{c}$ is maximized
But what about dynamics?
If the economy doesn't start at $\overline{K}$, does it ever get there?

**Short answer:** Yes, the economy always converges to $\overline{K}$ (as long as $K_0 > 0$).

However, it takes an infinite amount of time.
When $K < \bar{K}$, $sk^\alpha - K > 0$, so $K$ increases.
When $K > \bar{K}$, $sk^\alpha - K < 0$, so $K$ decreases.
Dynamics using algebra

\[ K_1 = s K_0^\alpha \]
\[ K_2 = s K_1^\alpha \]
\[ K_3 = s K_2^\alpha \]

etc.

\[ C_0 = Y_0 - I_0 = K_0^\alpha - K_1 \]
\[ C_1 = Y_1 - I_1 = K_1^\alpha - K_2 \]
\[ C_2 = Y_2 - I_2 = K_2^\alpha - K_3 \]

etc.

“Eventually”, both \( K_t \) and \( C_t \) converge to their steady-state values \( \bar{K} \) and \( \bar{C} \).
An important theoretical discovery:
Growth in the long run is ZERO!
The savings rate does NOT affect growth in the long run (that is, after the economy converges to its steady state).
Increases in the savings rate DO affect growth in the SHORT RUN but NOT in the LONG RUN.
An Increase in the Savings Rate

When $s$ increases, the economy moves to a new higher steady state.
An Improvement in Technology

$I = sA'K^\alpha$
$(A' > A)$

$sA'K^\alpha - K > 0$

45°

When technology improves (from $A$ to $A'$), the economy moves to a new higher steady state.
ANOTHER important theoretical discovery:
Sustained increases in technology lead to sustained increases in output, consumption, and the capital stock.

Improvements in technology overcome the problem of diminishing returns to capital.
This is what Sergey Brin means by “building ladders to larger, higher-hanging fruit.”