Welfare

1 Welfare in models with multiple agents

We are interested in analyzing the neoclassical growth model in an environment with multiple agents. Our objective will be to study the efficiency properties of competitive equilibrium under such setups. Will the introduction of several, possibly non-homogeneous, agents substantially modify the welfare properties of equilibria?

Heterogeneity among agents can be modelled by assuming agents with different preferences, or by endowing them with varying amounts of initial wealth. Another way of thinking of a non-homogeneous population is by using the "Overlapping Generations" approach, as opposed to the "Dynastic" approach we have been dealing with so far. The dynastic model has all its agents "born" in period \( t = 0 \), thereafter to live during all the periods modelled (possibly infinite time). On the contrary, the overlapping generations scheme assumes that new agents are born into the economy each period, and these individuals' life spans are shorter than the economy's time horizon.

The overlapping generations approach seems more realistic, and raises issues that the dynastic model does not address, such as why agents would "die" leaving non-consumed savings behind. In fact, a vast literature on the intergenerational game has developed, trying to explain the motives for bequests, and studying the impact of this game on the dynamic properties of the economy.

In what follows, we will introduce some general definitions. Let us first assume that there is a finite set \( H \) of consumers (and, abusing notation slightly, let \( H \) be an index set, such that \( H \equiv \text{card}(H) \)), then we can index individuals by a subscript \( h = 1, \ldots, H \). So \( H \) agents are born each period \( t \), and they all die at \( t + 1 \). Therefore, in each period \( t \) the young generation born at \( t \) lives together with the "old" people born at \( t − 1 \).

Let \( c^h_{t+i} \) denote consumption at date \( t+i \) of agent \( h \) born at \( t \) (usually we say "of generation \( t\)"), and we have the following:

Definition 1 A consumption allocation is a sequence

\[
c = \left\{ \left( c^h_t(t), c^h_{t+1}(t+1) \right)_{h \in H} \right\}_{t=1}^{\infty} \cup \left( c^h_0(1) \right)_{h \in H}
\]

Let \( c(t) \equiv \sum_{h \in H} \left[ c^h_t(t) + c^h_{t-1}(t) \right] \) denote total consumption at period \( t \), composed by the amount \( c^h_t(t) \) consumed by the young agents born at \( t \), and the consumption \( c^h_{t-1}(t) \) enjoyed by the old agents born at \( t − 1 \). Then we have the following:
Definition 2 (Feasibility) A consumption allocation is feasible if
\[ c(t) \leq Y(t) \quad \forall t \]

Example 3 (Storage economy) Assume there is "inter-temporal production" modelled as a storage technology whereby investing 1 unit at \( t \) yields \( \gamma \) units at \( t + 1 \). In this case, application of the previous definition reads:

A consumption allocation is feasible if there exists a sequence \( \{K(t)\}_{t=0}^{\infty} \) such that
\[ c(t) + K(t + 1) \leq Y(t) + K(t) \quad \forall t \]
where \( Y(t) \) is an endowment process.

Example 4 (Neoclassical growth model) Let \( L(t) \) be total labor supply at \( t \), and the neoclassical function \( Y(t) \) represent production technology:
\[ Y(t) = F[K(t), L(t)] \]

Capital is accumulated according to the following law of motion:
\[ K(t + 1) = (1 - \delta) \cdot K(t) + I(t) \]

Then in this case (regardless of whether this is a dynastic or an overlapping generations setup), we have that a consumption allocation is feasible if there exists a sequence \( \{I(t)\}_{t=0}^{\infty} \) such that
\[ c(t) + I(t) \leq F[K(t), L(t)] \quad \forall t \]

The definitions introduced so far are of a physical nature: they refer only to the material possibility to attain a given consumption allocation. We may also want to open judgement on the desirability of a given allocation. Economists have some notions that accommodate this need, and to that end we introduce the following:

Definition 5 (Efficiency) A feasible consumption allocation \( c \) is efficient if there is no alternative feasible allocation \( \tilde{c} \) such that
\[ \tilde{c}(t) \geq c(t) \quad \forall t \]
but \( \tilde{c}(t) > c(t) \) for some \( t \)

An allocation is thus deemed efficient if resources are not wasted; that is, if there is no way of increasing the total amount consumed in some period without decreasing consumption in the remaining periods.

The previous definition, then, provides a tools for judging the "desirability" of an allocation according to the aggregate consumption pattern. The following two definitions allow an extension of economists’ ability to assess this desirability to the actual distribution of goods among agents.
Definition 6 (Pareto domination) A feasible consumption allocation \( c_A \) is Pareto superior to \( c_B \) (or \( c_A \) "Pareto dominates" \( c_B \)) if:

1. No agent prefers the consumption path specified by \( c_B \) to that specified by \( c_A \):
   \[ c_A \succ_h c_B \quad \forall h \in H \]

2. At least one agent strictly prefers the allocation \( c_A \) to \( c_B \):
   \[ \exists j \in H : c_A \succ_j c_B \]

(Notice that this general notation allows each agent’s preferences to be defined on other agents’ consumption, as well as on his own.)

Whenever \( c_B \) is implemented, the existence of \( c_A \) implies that a welfare improvement is feasible by modifying the allocation. Notice that a welfare improvement in this context means that it is possible to provide at least one agent (and potentially many of them) with a consumption pattern that he will find preferable to the status quo, while the remaining agents will find the new allocation at least as good as the previously prevailing one.

Building on the previous definition, we can introduce one of economists’ most usual notions of the most desirable allocation that can be achieved in an economy:

Definition 7 (Pareto optimality) A consumption allocation \( c \) is Pareto Optimal if:

1. It is feasible.

2. There is no other feasible allocation \( \tilde{c} \neq c \) that Pareto dominates \( c \).

1.1 Overlapping generations

Even though we accommodated notation to suit the overlapping generations framework, the previous definitions are also extensive for the dynastic setup. In what follows we will restrict our attention to the overlapping generations model to study the efficiency and optimality properties of competitive equilibria. You may suspect that the fact that agents’ life spans are shorter than the economy’s horizon might lead to a different level of capital accumulation than if agents lived forever. In effect, a quite general result is that economies in which generations overlap lead to an overaccumulation of capital. This is a form of (dynamic) inefficiency, since an overaccumulation of capital implies
that the same consumption pattern could have been achieved with less capital investment - hence more goods could have been "freed-up" to be consumed.

In what follows, we will extend the concept of competitive equilibrium to the overlapping generations setup. We will start by considering endowment economies, thereafter to extend the analysis to production economies, and finally to the neoclassical growth model.

1.1.1 Endowment economy

We continue to assume that agents of every generation are indexed by the index set $H$. Let $\varpi^h_t(t+i)$ denote the endowment of goods at $t+i$ of agent $h$ born at $t$. Then the total endowment process is given by

$$Y(t) = \sum_{h \in H} \varpi^h_t(t) + \varpi^h_{t-1}(t)$$

We will assume throughout that preferences are strongly monotone - so all inequality constraints on consumption will bind.

Sequential markets We assume that contracts between agents specifying one-period loans are enforceable, and we let $R(t)$ denote the gross interest rate of loans granted at period $t$ and maturing at $t+1$. Then each agent $h$ born at $t \geq 1$ must solve:

$$\max_{c_1, c_2} u^h_h(c_1, c_2) \quad \text{(UtMax t)}$$

s.t. $c_1 + l \leq \varpi^h_t(t)$

$$c_2 \leq \varpi^h(t+1) + l \cdot R(t)$$

And generation 0 solves

$$\max_{c_0(1)} u^h_0\left[c_0^h(1)\right] \quad \text{(UtMax 0)}$$

s.t. $c_0^h(1) \leq \varpi^h_0(1)$

Unlike the dynastic case, there is no need for a no-Ponzi game restriction. In the dynastic model, agents could keep on building debt forever, unless prevented to do so. But now, they must repay their loans before dying, which happens in finite time (notice that in fact both the no-Ponzi-game and this "pay-before-you-die" restrictions are of an institutional nature, and they play a key role in the existence of an inter-temporal market - the credit market).

Definition: A competitive equilibrium is a consumption allocation $c$ and a sequence $R = \{R(t)\}_{t=0}^\infty$ such that:
1. \((c^h_t(t), c^h_t(t+1))\) solve generation \(t\)'s agent \(h\) (UtMax \(t\)) problem, and 
\(c^h_0(1)\) solves (UtMax 0) problem.

2. Market clearing is satisfied.

We need to require the credit market to be cleared:

\[
\sum_{h \in H} l^h_t = 0
\]

and Walras’ law will do the rest (since \(c\) must be feasible).

Therefore, there is no intergenerational borrowing in the endowment economy.

**Arrow-Debreu date-1 trading**  In this setup we assume that all future generations get together at date \(t = 1\) in a futures market and arrange delivery of consumption goods for the periods when they will live (you may assume that they all sign their trading contracts at \(t\), thereafter to die immediately and be reborn in their respective periods - the institutional framework in this economy allows enforcement of contracts signed in previous lives).

The futures market to be held at \(t\) will produce a price sequence \(\{p(t)\}_{t=1}^\infty\) of future consumption goods. Then each consumer (knowing in advance the date when he will be reborn to enjoy consumption) solves:

\[
\max_{c_1, c_2} u^h_t(c_1, c_2) \quad \text{(UtMax } t\text{)}
\]

\[
s.t. \ p(t) \cdot c_1 + p(t+2) \cdot c_2 \leq p(t) \cdot \varpi^h_t(t) + p(t+2) \cdot \varpi^h_t(t)
\]

whenever his next life will take place at \(t \geq 1\), and the ones to be born at \(t = 0\) will solve

\[
\max_c u^h_0(c) \quad \text{(UtMax } 0\text{)}
\]

\[
s.t. \ p(1) \cdot c \leq p(1) \cdot \varpi^h_0(1)
\]

**Definition:** A competitive equilibrium is a consumption allocation \(c\) and a sequence \(p \equiv \{p(t)\}_{t=0}^\infty\) such that:

1. \((c^h_t(t), c^h_t(t+1))\) solve generation \(t\)'s agent \(h\) (UtMax \(t\)) problem, and 
\(c^h_0(1)\) solves (UtMax 0) problem.

2. Resource feasibility is satisfied (markets clear).
Claim 8 The definitions of equilibrium with sequential markets and with Arrow-Debreu date-1 trading are equivalent. Moreover, if \((c, p)\) is an Arrow-Debreu date-1 trading equilibrium, then \((c, R)\) is a sequential markets equilibrium where

\[
R(t) = \frac{p(t)}{p(t+1)}
\]

Proof. Recall the sequential markets budget constraint of an agent born at \(t\):

\[
c_1 + l = \varpi^h_t(t),
\]

\[
c_2 = \varpi^h_t(t+1) + l \cdot R(t)
\]

where we use the strong monotonicity of preferences to replace the inequalities by equalities. Solving for \(l\) and replacing we obtain:

\[
c_1 + \frac{c_2}{R(t)} = \varpi^h_t(t) + \frac{\varpi^h_t(t+1)}{R(t)}
\]

Next recall the Arrow-Debreu date-1 trading budget constraint of the same agent:

\[
p(t) \cdot c_1 + p(t+1) \cdot c_2 = p(t) \cdot \varpi^h_t(t) + p(t+1) \cdot \varpi^h_t(t+1)
\]

Dividing through by \(p(t)\),

\[
c_1 + \frac{p(t+1)}{p(t)} \cdot c_2 = \varpi^h_t(t) + \frac{p(t+1)}{p(t)} \cdot \varpi^h_t(t+1)
\]

An identical argument shows that if \((c, R)\) is a sequential markets equilibrium, then \((c, p)\) is an Arrow-Debreu date-1 trading equilibrium, where prices \(p(t)\) are determined by normalizing \(p(1) = 1\) and (inductively) deriving the remaining ones from

\[
p(t+1) = \frac{p(t)}{R(t)}
\]

Remark 9 The equivalence of the two equilibrium definitions requires that the amount of loans that can be drawn, \(l\), be unrestricted (that is, that agents face no borrowing constraints other than the ability to repay their debts). The reason is that we can switch from

\[
c_1 + l = \varpi^h_t(t),
\]

\[
c_2 = \varpi^h_t(t+1) + l \cdot R(t)
\]

\begin{equation}
(BC \ 1)
\end{equation}

to

\[
c_1 + \frac{c_2}{R(t)} = \varpi^h_t(t) + \frac{\varpi^h_t(t+1)}{R(t)}
\]

\begin{equation}
(BC \ 2)
\end{equation}
only in the absence of any such restrictions.

Suppose instead that we had the added requirement that \( l \geq b \) for some number \( b \) such that \( b > -\frac{\varpi_t^h(t+1)}{R(t)} \). In this case, (BC 1) and (BC 2) would not be identical any more.

(If \( b = -\frac{\varpi_t^h(t+1)}{R(t)} \), then this is just the "pay-before-you-die" restriction - implemented in fact by non-negativity of consumption. Also, if \( b < -\frac{\varpi_t^h(t+1)}{R(t)} \), then \( l \geq b \) would never bind, for the same reason.).

**Application: Endowment Economy with One Agent per Generation**

We will assume that \( H = 1 \) (therefore agents are now in fact indexed by their birth dates), and that for every generation \( t \), preferences are represented by the following utility function:

\[
u_t(c_1, c_2) = \log c_1 + \log c_2
\]

The endowment processes are given by:

\[
\varpi_t(t) = \varpi_1
\]
\[
\varpi_t(t+1) = \varpi_2
\]

for all \( t \). Trading is sequential, and there are no borrowing constraints other than solvency.

Agent \( t \) now solves

\[
\max_{c_1, c_2} \log c_1 + \log c_2
\]

s.t. \( c_1 + \frac{c_2}{R(t)} = \varpi_1 + \frac{\varpi_2}{R(t)} \)

We can replace to transform the agent’s problem into:

\[
\max_{c_1, c_2} \log \left[ \left( \varpi_1 + \frac{\varpi_2}{R(t)} - c_1 \right) \cdot R(t) \right]
\]

First order condition:

\[
\frac{1}{c_1} - \frac{R(t)}{\left( \varpi_1 + \frac{\varpi_2}{R(t)} - c_1 \right) \cdot R(t)} = 0
\]

\[
c_1 = \varpi_1 + \frac{\varpi_2}{R(t)} - c_1
\]
Then
\[ c_1 = \frac{1}{2} \cdot \left( \varpi_1 + \frac{\varpi_2}{R(t)} \right) \]
\[ c_2 = \frac{1}{2} \cdot (\varpi_1 \cdot R(t) + \varpi_2) \]

Market clearing requires that the initial old consume exactly their endowment:
\[ c_0(1) = \varpi_2 \]

Therefore from the feasibility constraint for period \( t = 1 \), that reads:
\[ c_1(1) + c_0(1) = \varpi_1 + \varpi_2 \]
follows:
\[ c_1(1) = \varpi_1 \]
(Notice that the same result follows from clearing of the loans market at \( t = 1 \): \( l_1 = 0 \). This, together with \( c_1(1) + l_1 = \varpi_1 \), implies the same period 1 allocation.)

Repeating the market clearing argument for the remaining \( t \) (since \( c_1(1) = \varpi_1 \) will imply \( c_1(2) = \varpi_2 \)), we obtain the following equilibrium allocation:
\[ c_t(t) = \varpi_1 \]
\[ c_t(t + 1) = \varpi_2 \]
for all \( t \geq 1 \). Given this allocation, we solve for the prices \( R(t) \) that support it. You may check that these are:
\[ R(t) = \frac{\varpi_2}{\varpi_1} \]

This constant sequence supports the equilibrium where agents do not trade: they just consume their initial endowments.

Let us use specific numbers to analyze a quantitative example. Let
\[ \varpi_1 = 3 \]
\[ \varpi_2 = 1 \]

This implies the following \textit{gross} interest rate
\[ R(t) = \frac{1}{3} \]

The \textit{net} interest rate is negative: \( r(t) = R(t) - 1 = -\frac{2}{3} \). Recall the dynastic model. The equilibrium return on savings in that setup was \( R(t) = \beta^{-1} \). Hence each modelling techniques produces a different allocation.

The natural question, hence, is whether the outcome \( R(t) = \frac{1}{3} \) is a) Efficient; and b) Optimal.
a) Efficiency: Total consumption under the proposed allocation is \(c(t) = 4\); and this is equal to the total endowment. Hence it is not possible to increase consumption in any period; there is no waste of resources. Therefore the allocation is efficient.

b) To check whether the allocation is optimal, consider the following alternative allocation:

\[
\begin{align*}
\tilde{c}_0(1) &= 2 \\
\tilde{c}_t(t) &= 2 \\
\tilde{c}_t(t+1) &= 2
\end{align*}
\]

That is, the allocation \(\tilde{c}\) obtains from a chain of inter-generational good transfers that consists of the young in every period giving a unit of their endowment to the old at every period. Notice that for all generations \(t \geq 1\), this is just a modification of the timing in their consumption, since total goods consumed throughout their lifetime remain at 4. For the initial old, this is an increase from 1 to 2 units of consumption when old. It is clear, then, that the initial old strictly prefer \(\tilde{c}\) to \(c\). We need to check what the remaining generations think about the change. It is clear that since utility is concave (the log function is concave), this even split of the same total amount will yield a higher utility value. In effect:

\[
\begin{align*}
\log 2 + \log 2 &= 2 \\
\log 4 &= \log 3
\end{align*}
\]

Therefore, \(\tilde{c}\) Pareto dominates \(c\); hence \(c\) cannot be optimal.

Suppose instead that the endowment process is reversed in the following way:

\[
\begin{align*}
\varpi_1 &= 1 \\
\varpi_2 &= 3
\end{align*}
\]

There is the same total endowment in the economy each period, but the relative assignments of young and old are reversed. From the formula that we have derived above, this implies

\[
R(t) = 3
\]

The "no trade" equilibrium where each agent consumes his own endowment each period is efficient again, since no goods are wasted. Is it Pareto optimal? This seems a difficult issue to address, since we need to compare the prevailing allocation with all other possible allocations. We already know that an allocation having \((2, 2)\) will be preferred to \((1, 3)\) given the log utility assumption. However, is it possible to unchain a sequence of inter-generational transfers achieving consumption of \((2, 2)\) from some \(t (\geq 1)\) onwards, while keeping the constraint that all generations \(t \geq 1\) consume at least 4 units throughout their
lifetime, and the initial old consume 3 units? (Otherwise, if any of these two constraints is violated, the allocation thus obtained will not Pareto dominate the "no trade" allocation.)

Next notice that in analyzing Pareto optimality, we have restricted our attention to stationary allocations. Let us introduce a more formal definition of this term:

**Definition 10 (Stationary allocation)** A feasible allocation \( c \) is called stationary if \( \forall t \geq T : \)

\[
\begin{align*}
  c_t(t) &= c_1 \\
  c_t(t + 1) &= c_2
\end{align*}
\]

[Check this definition. Is it correct? What if the endowment process is not stationary? - Suggestion: Move this definition to the initial section of definitions]

With this definition at hand, we can pose the question of whether there is any stationary allocation that Pareto dominates \((2, 2)\). The following chart shows the resource constraint of the economy, plotted together with the utility level curve corresponding to the allocation \((2, 2)\):

The shaded area is the feasible set, its frontier given by the line \( c_y + c_o = 4 \). It is clear from the tangency at \((2, 2)\) that it is not possible to find an alternative allocation that Pareto dominates this one. However, what happens if we widen our admissible range of allocations and think about non-stationary ones? Could there be a non-stationary allocation dominating \((2, 2)\)?

In order to implement such a stationary allocation, a chain of inter-generational transfers would require a transfer from young to old at some arbitrary point in
time \( t \). These agents giving away endowment units in their youth would have to be compensated when old. The question is how many goods would be required for this compensation.

The chart illustrates that given an initial transfer \( \varepsilon_1 \) from young to old at \( t \), the transfer \( \varepsilon_2 \) required to compensate generation \( t \) must be larger than \( \varepsilon_1 \), given the concave utility assumption. This in turn will command a still larger \( \varepsilon_3 \), and so on. Is the sequence \( \{\varepsilon_t\}_{t=0}^{\infty} \) thus formed feasible?

An heuristic answer can be seen in the chart: no such a transfer scheme is feasible in the long run. Therefore, for this type of preferences the stationary allocation \((2, 2)\) is the Pareto optimal allocation. Any proposed non-stationary allocation that Pareto dominates \((2, 2)\) becomes unfeasible at some point in time.

Somewhat more formally, let us try to use the First Welfare Theorem to prove Pareto optimality. Notice that our model satisfies the following two key assumptions:

1. Preferences exhibit local non-satiation (since \( u \) is strictly increasing).
2. The market value of all goods is finite (we will come back on this).

Then we can construct the following proof:  
(Suggestion: move this proof to the section where definitions are introduced.)

**Proof** (Pareto optimality of competitive equilibrium). Let an economy’s population be indexed by a countable set \( I \) (possibly infinite), and consider a competitive equilibrium allocation \( x \) that assigns \( x_i \) to each agent \( i \) (\( x_i \) might be multi-dimensional).

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If $x$ is not Pareto optimal, then there exists $\hat{x}$ that Pareto dominates $x$, that is, a feasible allocation that verifies:

$$\forall i \in I : \hat{x}_i \succeq_i x_i$$
$$\exists j \in I : \hat{x}_j >_j x_j$$

Then we can use local non-satiation to show that

$$p \cdot \hat{x}_i \geq p \cdot x_i$$
$$p \cdot \hat{x}_j > p \cdot x_j$$

must hold.

Summing up over all agents,

$$\sum_{i \in I} p \cdot \hat{x}_i > \sum_{i \in I} p \cdot x_i$$

And the last inequality violates the market clearing condition, since the market value of goods (with local non-satiation) must be equal to the market value of endowments in an equilibrium. You may observe that this proof is in fact an application of the separating hyperplane theorem.

This proof is quite general. In the specific case of overlapping generations, we have the following two peculiarities:

1. $p$ and $x$ are infinite-dimensional vectors.
2. There is an infinity of consumers.

Therefore the series $\sum_{i \in I} p \cdot \hat{x}_i$ and $\sum_{i \in I} p \cdot x_i$ might take infinite value, in which case the last comparison in the proof might not hold. We need to specify further conditions to ensure that the first welfare theorem will hold, even with the "correct" assumptions on preferences.

To this effect, let us assume that the following are met by an economy:

1. Regularity conditions on utility and endowments.
2. Restrictions on the curvature below of the utility function - that has to be "somewhat" curved, but not too much. An example of curvature measure is the inter-temporal elasticity of substitution:

$$\frac{f''(x) \cdot x}{f'(x)}$$
(Yes, also called the coefficient of relative risk aversion whenever the environment involves uncertainty - the same ratio measures two aspects of preferences: inter-temporal comparison, and degree of dislike for stochastic variability of consumption.)

3. Other technical details that you may find in the corresponding paper, but that are beyond the scope of this course.

Then we have the following:

**Theorem 11 (Balasko and Shell, Journal of Economic Theory, 1980)**

*A competitive equilibrium in an endowment economy populated by overlapping generations of agents is Pareto optimal if and only if*

\[ \sum_{t=1}^{\infty} \frac{1}{p(t)} = \infty \]

Recall our example. The allocation \((2, 2)\) implied \(R(t) = 1\), and from the equivalence of sequential and Arrow-Debreu date-1 trading equilibria, we have that

\[ p(t + 1) = \frac{p(t)}{R(t)} \]

Therefore

\[ \sum_{t=1}^{\infty} \frac{1}{p(t)} = \sum_{t=1}^{\infty} \frac{1}{p(1)} = \infty \]

In the case of \((3, 1)\), we have that

\[ p(t + 1) = 3^t \cdot p(1) \]

then

\[ \sum_{t=1}^{\infty} \frac{1}{p(t)} = \sum_{t=1}^{\infty} \frac{3^{-t}}{p(1)} = \frac{1}{p(1)} \cdot \sum_{t=1}^{\infty} 3^{-t} = \frac{1}{2} \cdot \frac{1}{p(1)} < \infty \]

And finally for \((1, 3)\),

\[ \sum_{t=1}^{\infty} \frac{1}{p(t)} = \sum_{t=1}^{\infty} \frac{3^t}{p(1)} = \infty \]

Therefore by applying the theorem we conclude that \((2, 2)\) and \((1, 3)\) are Pareto optimal allocations, whereas \((3, 1)\) can be improved upon - the same conclusion we had reached before.
1.1.2 Economies with inter-temporal assets

In the previous section, we have looked at overlapping generations economies in which only consumption goods are traded. A young agent selling part of his endowment to an old one needs of course something which serves the purpose of storage of value, so that the proceeds from the sale performed at time \( t \) can be used to purchase goods at \( t + 1 \). Money is therefore implicit in the framework of the previous section, which is obvious from the moment that such thing as "prices" are mentioned.

However, notice that such money is just "fiat" money. Any paper with a number printed on it will fulfill the need of value storage, provided that everybody agrees on which are the valid papers, and no forgery occurs. We have assumed away these details: agents are honest.

We will now introduce assets into the economy (in addition to fiat money). Rather than fiat money, assets will be real claims; that is, actual rights to receive goods in the following periods. Two different kinds of assets are of interest:

- A tree that produces a given fruit yield (dividend) each period.
- Capital, that can be used to produce goods with a given technology.

**A tree economy** We assume that the economy is populated by one agent per generation, and that each agent lives for two periods. Preferences are represented by a logarithmic utility function as in previous examples:

\[
u_t (c^1_t, c^2_t) = \log c^1_t + \log c^2_t\]

Agents are endowed with \((\varpi_1, \varpi_2)\) consumption units (fruits) when young and old, respectively, and there is also a tree that produces a fruit yield of \( d \) units each period. Therefore total resources in the economy each period are given by:

\[Y(t) = \varpi_1 + \varpi_2 + d\]

Ownership of a given share in the tree gives the right to collect such share out of the yearly fruit produce. Trading of property rights on the tree is enforceable, so any agent that finds himself owning any part of the tree when old will be able to sell it to the young in exchange for consumption goods. The initial old owns 100% of the tree.

Let \( a_{t+1} \) denote the share of the tree purchased by the young generation at \( t \). It is clear that asset market clearing requires \( a_{t+1} = 1 \) for all \( t \). Generation \( t \)
consumer solves:

$$\max_{c_1^t, c_2^t} \log c_1^t + \log c_2^t$$

s.t. $$p_t \cdot a_{t+1} + c_1^t = \varpi_1$$

$$c_2^t = \varpi_2 + a_{t+1} \cdot (p_{t+1} + d)$$

Notice that the returns on savings are given by

$$\frac{p_{t+1} + d}{p_t}$$

The first order conditions yield

$$c_1^t = \frac{1}{2} \left( \varpi_1 + \frac{p_t}{p_{t+1} + d} \cdot \varpi_2 \right)$$

Which implies that generation $$t$$’s saving satisfy:

$$p_t \cdot a_{t+1} = \frac{1}{2} \left( \varpi_1 - \frac{p_t}{p_{t+1} + d} \cdot \varpi_2 \right)$$

Imposing the market clearing condition and rearranging we get the law of motion for prices:

$$p_{t+1} = \frac{\varpi_2}{\varpi_1} \cdot \frac{1}{2} - d \frac{p_t}{p_{t+1} + d}$$

This is a first order (non-linear) difference equation in $$p_t$$. The following chart shows that it has two fixed points, the stable one negative and the unstable one positive.
Then what is the equilibrium \( \{p_t\}_{t=1}^{\infty} \) sequence? It must be a constant sequence since any deviation from the positive fixed point leads directly into the negative one. \( p_t = p^* \forall t \), where \( p^* \) is the positive solution to

\[
p^* = \frac{\varpi_2}{\varpi_1 - 2} - d
\]

Is this competitive equilibrium Pareto optimal? We can answer this question by checking whether the Balasko-Shell criterion is satisfied. First notice that if we multiply \( \frac{1}{p_t} \) by \( \frac{p_{t-1} \cdot p_{t-2} \cdots p_2 \cdot p_1}{p_{t-1} \cdot p_{t-2} \cdots p_2 \cdot p_1} \), we can write:

\[
\frac{1}{p_t} = \frac{p_{t-1} \cdot p_{t-2} \cdots p_2 \cdot p_1}{p_1 \cdot p_{t-1} \cdots p_2 \cdot p_1} = \prod_{s=1}^{t-1} R_{s, s+1}
\]

where \( R_{s, s+1} \) denotes the interest rate between periods \( s \) and \( s + 1 \):

\[
R_{s, s+1} = \frac{p_s}{p_{s+1}}
\]

But we already know that the return on savings is given by:

\[
\frac{p_{t+1} + d}{p_t}
\]

So the interest rate for each period, using equilibrium prices, is

\[
R_{s, s+1} = \frac{p^* + d}{p^*}
\]

Therefore, replacing for \( \frac{1}{p_t} \), we get that:

\[
\sum_{t=1}^{\infty} \frac{1}{p_t} = p_1 \cdot \sum_{t=1}^{\infty} \left(1 + \frac{d}{p^*}\right)^t
\]

And the limit of this series is \( \infty \) for any \( d \geq 0 \). The Balasko-Shell criterion is met; hence the competitive equilibrium allocation supported by these prices is Pareto optimal.

Finally, notice that the optimality of the result was proven regardless of the actual endowment process; hence it generalizes for any possible such process.
A production economy We will assume the simplest form of production, namely constant marginal returns on capital. Such a technology, represented by a linear function on capital, is what we have called "storage" technology whenever no labor inputs are needed in the production process. Let the yield obtained from storing one unit be equal to one. That is, keeping goods for future consumption involves no physical depreciation, nor increases the physical worth of the stored goods.

Let the marginal rates of substitution between consumption when old and when young be captured by a logarithmic function, as before, and assume that the endowment process is \((\varpi_1, \varpi_2) = (3, 1)\). Generation \(t\)'s problem is therefore:

\[
\max_{c^t_y, c^t_o} \log c^t_y + \log c^t_o \\
\text{s.t. } s_t + c^t_y = \varpi_1 \\
\quad c^t_o = s_t + \varpi_2
\]

The first order conditions yield

\[
c^t_y = \frac{1}{2} \left( \varpi_1 + \frac{\varpi_2}{R_t} \right)
\]

The return on storage is one, \(R_t = 1\). So, using the values assumed for the endowment process, this collapses to

\[
c^t_y = 2 \\
c^t_o = 2 \\
s_t = 1
\]

Notice that the allocation corresponds to what we have found to be the Pareto optimal allocation before: \((2, 2)\) is consumed by every agent. However, in the previous case where no real inter-temporal assets existed in the economy, such an allocation was achieved by a chain of inter-generational transfers (enforced, if you like, by the exchange in each period of those pieces of paper dubbed fiat money). Whereas now each agent buries his potato when young, and consumes it when old.

Then, is the current allocation Pareto optimal? The answer is clearly no, since, to achieve the consumption pattern \((2, 2)\), the potato must always be buried on the ground. The people who are born at \(t = 0\) set aside one unit of their endowment to consume when old, and thereafter all their descendence mimic this behavior, for a resulting allocation

\[
c = \{(2, 2)\}_{t=0}^{\infty}
\]

However, the following improvement could be implemented. Suppose that instead of storing one, the first generation \((t = 0)\) consumed its three units when
young. In the following period, the new young would, give them their own spare unit, instead of storing it, thereafter to continue this chain of intergenerational transfers through infinity and beyond. The resulting allocation would be:

\[ \hat{c} = (3, 2) \cup \{(2, 2)\}_{t=1}^\infty \]

A Pareto improvement on \( c \).

In fact, \( \hat{c} \) is not only a Pareto improvement on \( c \), but simply the same allocation \( c \) plus one additional consumption unit enjoyed by generation 0. Since the total endowment of goods is the same, this must mean that one unit was being wasted under allocation \( c \).

This problem is called in the literature "overaccumulation of capital". The equilibrium outcome is not efficient.

**Neoclassical growth model** The production technology is now modelled by a neoclassical production function. Capital is owned by the old, who put it to production and then sell it to the young each period. Agents have a labor endowment of \( \varpi_y \) when young and \( \varpi_o \) when old. Assuming that leisure is not valued, generation \( t \)'s utility maximization problem is:

\[
\begin{align*}
\max_{c_y^t, c_o^t} & \quad \{ c_y^t, c_o^t \} \\
\text{s.t.} & \quad c_y^t + s_t = \varpi_y \cdot w_t \\
& \quad c_o^t = s_t \cdot r_{t+1} + \varpi_o \cdot w_{t+1}
\end{align*}
\]

If the utility function is strictly quasiconcave, the savings correspondence that solves this problem is single-valued:

\[ s_t = h[w_t, r_{t+1}, w_{t+1}] \]

The asset market clearing condition is:

\[ s_t = K_{t+1} \]

We require the young at \( t \) to save enough to purchase next period’s capital stock, which is measured in terms of consumption goods (the relative price of capital and consumption goods is 1).

The firm operates the production technology, that is represented by the function \( F(K, n) \). The market clearing condition for labor is

\[ n_t = \varpi_y + \varpi_o \]
From the firm's first order conditions for maximization, we have that factor remunerations are determined by

\[ r_t = F_1(K_t, \varpi_y + \varpi_o) \]
\[ w_t = F_2(K_t, \varpi_y + \varpi_o) \]

If we assume that the technology exhibits constant returns to scale, we may write

\[ F(K, n) = n \cdot f \left( \frac{K}{n} \right) \]

where \( f \left( \frac{K}{n} \right) \equiv F \left( \frac{K}{n}, 1 \right) \). Replacing in the expressions for factor prices,

\[ r_t = f' \left( \frac{K_t}{\varpi_y + \varpi_o} \right) \]
\[ w_t = f \left( \frac{K_t}{\varpi_y + \varpi_o} \right) - \frac{K_t}{\varpi_y + \varpi_o} \cdot f' \left( \frac{K_t}{\varpi_y + \varpi_o} \right) \]

Let \( k_t \equiv \frac{K_t}{\varpi_y + \varpi_o} \) denote the capital/labor ratio. If we normalize \( \varpi_y + \varpi_o = 1 \), we have that \( K_t = k_t \). Then

\[ r_t = f'(k_t) \]
\[ w_t = f(k_t) - k_t \cdot f'(k_t) \]

Substituting in the savings function, and imposing asset market equilibrium,

\[ k_{t+1} = h \left[ f(k_t) - k_t \cdot f'(k_t), f'(k_t), f(k_{t+1}) - k_{t+1} \cdot f'(k_{t+1}) \right] \]

We have obtained a first order differential equation. Recall that the dynastic model lead to a second order equation instead. However, proving convergence to a steady state is usually more difficult in the overlapping generations setup. Recall that the steady state condition with the dynastic scheme was of the form

\[ \beta \cdot f'(k^*) = 1 \]

In this case, steady state requires that

\[ k^* = h \left[ f(k^*) - k^* \cdot f'(k^*), f'(k^*), f(k^*) - k^* \cdot f'(k^*) \right] \]
1.2 Dynamic efficiency in models with multiple agents

We have analyzed the welfare properties of consumption allocations arising from a multiple agent environment under the form of a population consisting of overlapping generations of individuals. The purpose of this section is to generalize the study of the dynamic efficiency of an economy to a wider range of modelling assumptions. In particular, we will present a theorem valid for any form of one-sector growth model.

We assume that the technology is represented by a neoclassical production function that satisfies the following properties:

- \( f(0) = 0 \)
- \( f'(\cdot) > 0 \)
- \( f''(\cdot) < 0 \)
- \( f \in C^1 \) (\( C^1 \) denotes the space of continuously differentiable functions)
- \( \lim_{x \to 0} f'(x) = \infty \)
- \( \lim_{x \to \infty} f'(x) = 0 \)
- I think that \( \delta = 1 \) assumption is missing, or otherwise the theorem below should read \( f'(k^*) \geq \delta. \)

Then we can show the following:

**Theorem 12** A steady state \( k^* \) is efficient if and only if \( R^* \equiv f'(k^*) \geq 1 \).

Intuitively, the steady state consumption is \( c^* = f(k^*) - k^* \) (since there is full depreciation). The chart below shows the attainable levels of steady state capital stock and consumption \( (k^*, c^*) \) attainable, given the assumptions on \( f \). The \( (k^G, c^G) \) locus corresponds to the "golden rule" level of steady state capital and consumption, that maximize \( c^G \).
Proof. (i) $R^* < 1$ is inefficient.

Assume that $k^*$ is such that $f'(k^*) < 1$. Let $c^*$ denote the corresponding level of steady state consumption, let $c_0 = c^*$. Now consider a change in the consumption path, whereby $k_1$ is set to $k_1 = k^* - \varepsilon$ instead of $k_1 = k^*$. Notice this implies an increase in $c_0$. Let $k_t = k_1 \forall t \geq 1$. We have that

$$c_1 - c^* = f(k_1) - k_1 - f(k^*) + k^*$$

Notice that strict concavity of $f$ implies that

$$f(k^*) < f(k^* - \varepsilon) + [k^* - (k^* - \varepsilon)] \cdot f'(k^* - \varepsilon)$$

for $\varepsilon \in (0, k^* - k^G)$, we have that $f'(k^* - \varepsilon) < 1$. Therefore,

$$f(k^*) < f(k^* - \varepsilon) + k^* - (k^* - \varepsilon)$$

This implies that

$$c_1 - c^* > 0$$

Which shows that a permanent increase in consumption is feasible.

(ii) $R^* \geq 1$ is efficient.

Suppose not, then we could decrease the capital stock at some point in time and achieve a permanent increase in consumption (or at least increase consumption at some date without decreasing consumption in the future). Let the initial situation be a steady state level of capital $k_0 = k^*$ such that $f'(k^*) \geq 1$. Let the initial $c_0$ be the corresponding steady state consumption: $c_0 = c^* = f(k^*) - k^*$. Since we suppose that $k^*$ is inefficient, consider a decrease of capital
accumulation at time 0: \( k_1 = k^* - \varepsilon \), thereby increasing \( c_0 \). We need to maintain the previous consumption profile \( c^* \) for all \( t \geq 1 \): \( c_t \geq c^* \). This requires that

\[
\begin{align*}
c_1 &= f(k_1) - k_2 \geq f(k^*) - k^* = c^* \\
k_2 &\leq f(k_1) - f(k^*) + k^* \\
\frac{k_2 - k^*}{\varepsilon_2} &\leq f(k_1) - f(k^*)
\end{align*}
\]

And concavity of \( f \) implies that

\[
f(k_1) - f(k^*) < f'(k^*) \cdot \left[ k_1 - k^* \right]
\]

Notice that \( \varepsilon_2 = k_2 - k^* < 0 \). Therefore, since \( f'(k^*) \geq 1 \) by assumption, we have that \( |\varepsilon_2| > |\varepsilon_1| \)

The size of the decrease in capital accumulation is increasing. By induction, \( \{\varepsilon_t\}_{t=0}^{\infty} \) is a decreasing sequence (of negative terms). Since it is bounded below by \( -k^* \), we now from real analysis that it must have a limit point \( \varepsilon_\infty \in [-k^*, 0) \). Consequently, the consumption sequence converges as well:

\[
c_\infty = f(k^*) - \varepsilon_\infty - (k^* - \varepsilon_\infty)
\]

It is straightforward to show, using concavity of \( f \), that \( c_\infty < c^* \)

Then the initial increase in consumption is not feasible if the restriction is to maintain at least \( c^* \) as the consumption level for all the remaining periods of time.

We now generalize the theorem, dropping the assumption that the economy is in steady state.

**Theorem 13 (Dynamic efficiency with possibly non-stationary allocations)**

Let both \( \{k_t\}_{t=0}^{\infty} \) and the associated sequence \( \{R_t(k_t) \equiv f'_t(k_t)\}_{t=0}^{\infty} \) be bounded above and below away from zero. Let \( 0 < a \leq -f''_t(k_t) \leq M < \infty \ \forall t, \forall k_t \).

Then \( \{k_t\}_{t=0}^{\infty} \) is efficient if and only if

\[
\sum_{t=0}^{\infty} \left[ \prod_{s=1}^{t} R_s(k_s) \right] = \infty
\]

Recall that

\[
\sum_{t=0}^{\infty} \left[ \prod_{s=1}^{t} R_s(k_s) \right] = \sum_{t=0}^{\infty} \frac{1}{p_t}
\]

The Balasko-Shell criterion discussed when studying overlapping generations is then a special case of the theorem just presented.
2 Welfare theorems in the dynastic model

From our discussion so far, we can draw the following summary conclusions on the applicability of the first and second welfare theorems to the dynamic economy model.

First Welfare Theorem

1. Overlapping Generations: Competitive equilibrium is not always Pareto optimal. Sometimes it is not even efficient.

2. Dynastic model: Only local non-satiation of preferences is required for competitive equilibrium to be Pareto optimal.

Second Welfare Theorem

1. Overlapping Generations: In general, there is no applicability of the Second Welfare Theorem.

2. Dynastic model: Only convexity assumptions are required for any Pareto optimal allocation to be implementable as a competitive equilibrium.

Therefore with the adequate assumptions on preferences and on the production technology, the dynastic model yields an equivalence between competitive equilibrium and Pareto optimal allocations. Of course, the restrictions placed on the economy for the Second Welfare Theorem to apply are much stronger than those required for the First one to hold. Local non-satiation is almost not an assumption in economics, but virtually the defining characteristic of our object of study (recall that phrase talking about scarce resources, etcetera).

In what follows, we will study the Second Welfare Theorem in the dynastic model. To that effect, we first study a 1-agent economy, and after that a 2-agents one.

2.1 The second welfare theorem in a 1-agent economy

We assume that the consumer’s preferences over infinite consumption sequences and leisure are represented by a utility function with the following form:

\[ U \left( \{c_t, l_t\}_{t=0}^\infty \right) = \sum_{t=0}^{\infty} \beta^t \cdot u(c_t) \]

where \( 0 < \beta < 1 \) and the felicity index \( u(\cdot) \) is strictly increasing and strictly concave. For simplicity, leisure is not valued.
This is a one-sector economy in which the relative prices of capital and consumption goods is 1. Production technology is represented by a concave, homogeneous of degree one function of the capital and labor inputs:

\[ Y(t) = F(K_t, n_t) \]

Then the central planner's problem is:

\[ V(K_0) = \max_{\{c_t, K_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t \cdot u(c_t) \right\} \]

s.t. \( c_t + K_{t+1} = F(K_t, n_t) \)

The solutions to this problem are the Pareto optimal allocations. Then suppose we have an allocation \( \{c^*_t, K_{t+1}^*, n_t\}_{t=0}^{\infty} \) solving this planner’s problem and we want to support it as a competitive equilibrium. Then we need to show that there exist sequences \( \{p^*_t\}_{t=0}^{\infty}, \{R^*_t\}_{t=0}^{\infty}, \{w^*_t\}_{t=0}^{\infty} \) such that:

(i) \( \{c^*_t, K_{t+1}^*, n_t\}_{t=0}^{\infty} \) maximizes consumer’s utility subject to the budget constraint determined by \( \{p^*_t, R^*_t, w^*_t\}_{t=0}^{\infty} \).

(ii) \( \{K^*_t, n_t\}_{t=0}^{\infty} \) maximize firm’s profits.

(iii) Markets clear (the allocation \( \{c^*_t, K_{t+1}^*\}_{t=0}^{\infty} \) is resource-feasible).

Note: Even though \( n_t \) can be treated as a parameter for the consumer’s problem, this is not the case for the firms. These actually choose their amount of labor input each period. Therefore, we must make the sequence \( n_t \) part of the competitive equilibrium, and require that the wage level for each \( t \) support this as firms’ equilibrium labor demand.

A straightforward way of showing that the sequences \( \{p^*_t\}_{t=0}^{\infty}, \{R^*_t\}_{t=0}^{\infty}, \{w^*_t\}_{t=0}^{\infty} \) exist is directly by finding their value. Notice that from concavity of \( F(\cdot, \cdot) \),

\[
R^*_t = F_1(K_t^*, n_t), \\
w^*_t = F_2(K_t^*, n_t)
\]

will ensure that firms maximize profits (or if you like, that the labor and capital services markets clear each period). In addition, homogeneity of degree 1 implies that these factor payments completely exhaust production, so that the consumer ends up receiving the whole produce obtained from his factor supply.

Then the values of \( p^*_t \) remain to be derived. Recall the first order conditions in the planner’s problem:

\[
\beta^t \cdot u'(c^*_t) = \lambda^*_t \\
\lambda^*_t = F_1(K_{t+1}^*, n_{t+1}) \cdot \lambda^*_{t+1}
\]
which lead to the centralized Euler equation

\[ u'(c_t^*) = \beta \cdot u'(c_{t+1}^*) \cdot F_1(K_{t+1}^*, n_{t+1}) \]  

(CEE)

Now, since \( \lambda_t^* \) is the marginal value of relaxing the planner’s problem resource constraint at time \( t \), it seems natural that prices in a competitive equilibrium must reflect this marginal value as well. That is, \( p_t^* = \lambda_t^* \) seems to reflect the marginal value of the scarce resources at \( t \). Replacing in the planner’s Euler equation, we get that

\[ F_1(K_{t+1}^*, n_{t+1}) = \frac{p_t^*}{p_{t+1}^*} \]

Replacing by \( R_t^* \), this reduces to

\[ R_t^* = \frac{p_t^*}{p_{t+1}^*} \]

(MEE)

It is straightforward to check that (MEE) is the market Euler equation that obtains from the consumer’s first order conditions in the decentralized problem (you should check this). Therefore these prices seem to lead to identical consumption and capital choices in both versions of the model. We need to check, however, that the desired consumption and capital paths induced by these prices are feasible; that is, that these are market clearing prices. To that effect, recall the planner’s resource constraint (which binds due to local non-satiation):

\[ c_t^* + K_{t+1}^* = F(K_{t+1}^*, n_{t+1}) \quad \forall t \]

The equality remains unaltered if we premultiply both sides by \( p_t^* \):

\[ p_t^* \cdot [c_t^* + K_{t+1}^*] = p_t^* \cdot F(K_{t+1}^*, n_{t+1}) \quad \forall t \]

And summing up over \( t \), we get:

\[ \sum_{t=0}^{\infty} p_t^* \cdot [c_t^* + K_{t+1}^*] = \sum_{t=0}^{\infty} p_t^* \cdot F(K_{t+1}^*, n_{t+1}) \]

Finally, homogeneity of degree 1 of \( F(\cdot, \cdot) \) and the way we have constructed \( R_t^* \) and \( w_t^* \) imply that

\[ \sum_{t=0}^{\infty} p_t^* \cdot [c_t^* + K_{t+1}^*] = \sum_{t=0}^{\infty} p_t^* \cdot [R_t^* \cdot K_t^* + w_t^* \cdot n_t] \]

Therefore the budget constraint in the market economy is satisfied if the sequence \( \{c_t^*, K_{t+1}^*, n_t, p_t^*, w_t^*, R_t^*\}_{t=0}^{\infty} \) is chosen whenever the prevailing prices are \( \{p_t^*, w_t^*, R_t^*\}_{t=0}^{\infty} \).

Next we need to check whether the conditions for \( \{c_t^*, K_{t+1}^*, n_t, p_t^*, w_t^*, R_t^*\}_{t=0}^{\infty} \) to be a competitive equilibrium are satisfied or not:
(i) Utility maximization subject to budget constraint: We have seen that the budget constraint is met. To check whether this is in fact a utility maximizing consumption-capital path, we should take first order conditions. But it is straightforward that these conditions lead to the Euler equation (MEE) which is met by the planner’s optimal path \( \{K_{t+1}^*\}_{t=0}^\infty \).

(ii) Firms’ maximization: By construction of the factor services prices, and concavity of the production function, we have that \( \{K_t^*, n_t\}_{t=0}^\infty \) are the firms’ profit maximizing levels of factor inputs.

(iii) Market clearing: We have discussed before that the input markets clear. And we have seen that if the consumer’s decentralized budget constraint is met, this implies that the planner’s problem resource constraint is met for the corresponding consumption and capital sequences. Therefore the proposed allocation is resource-feasible.

Recall we mentioned convexity as a necessary assumption for the Second Welfare Theorem to hold.

Convexity of preferences entered our previous proof in that the first order conditions were deemed sufficient to identify a utility maximizing consumption bundle.

Convexity of the consumption possibilities set took the form of a homogeneous of degree one, jointly concave function \( F \). Concavity was used to establish the levels of factor remunerations \( R_t^* \), \( w_t^* \) that support \( K_t^* \) and \( n_t \) as the equilibrium factor demand by taking first order conditions on \( F \). And homogeneity of degree one ensured that with \( R_t^* \) and \( w_t^* \) thus determined, the total product would get exhausted in factor payment - an application of the Euler Theorem.

### 2.2 The second welfare theorem in a 2-agent economy

We now assume an economy with the same production technology and inhabited by two agents. Each agent has preferences on infinite-dimensional consumption vectors represented by the function

\[
U_i [(c_{it})_{t=0}^\infty] = \sum_{t=0}^{\infty} \beta_i^t \cdot u_i (c_{it}) \quad i = 1, 2
\]

Where \( \beta_i \in (0, 1) \) and \( u_i (\cdot) \) is strictly increasing, concave, for both \( i = 1, 2 \).

For some arbitrary weights \( \mu_1, \mu_2 \), we define the following welfare function:

\[
W [(c_{1t})_{t=0}^\infty, (c_{2t})_{t=0}^\infty] = \mu_1 \cdot U_1 [(c_{1t})_{t=0}^\infty] + \mu_2 \cdot U_2 [(c_{2t})_{t=0}^\infty]
\]
Then the following welfare maximization problem can be defined:

\[
V(K_0) = \max_{\{c_t, c_{t+1}, K_{t+1}\}} \left\{ \mu_1 \cdot \sum_{t=0}^{\infty} \beta^t \cdot u_1(c_{1t}) + \mu_2 \cdot \sum_{t=0}^{\infty} \beta^t \cdot u_2(c_{2t}) \right\}
\]

s.t. \( c_{1t} + c_{2t} + K_{t+1} \leq F(K_t, n_t) \)

where \( n_t = n_{1t} + n_{2t} \) denotes the aggregate labor endowment, which is fully utilized for production since leisure is not valued.

If we restrict \( \mu_1 \) and \( \mu_2 \) to be nonnegative and to add up to 1 (then \( W \) is a convex combination of the \( U_i \)'s), we have the Neguishi characterization: By varying the vector \( (\mu_1, \mu_2) \), all the Pareto optimal allocations in this economy can be obtained from the solution of the problem \( V(K_0) \).

That is, for every pair \( (\mu, 1-\mu) \) with \( \mu \in [0, 1] \) \( \mu_1, \mu_2 \geq 0, \mu_1 + \mu_2 = 1 \), we obtain a Pareto optimal allocation by solving \( V(K_0) \). Now, given any such allocation \( (c^*_1, c^*_2, K^*_t)\), is it possible to decentralize the problem \( V(K_0) \) so as to obtain that allocation as a competitive equilibrium outcome? Will the necessary price sequences to support this as a competitive equilibrium exist?

In order to analyze this problem, we proceed as before. We look for the values of \( \{p^*_t, R^*_t, w^*_t\}\) and we guess them using the same procedure:

\[
\begin{align*}
p^*_t &= \lambda^*_t \\
R^*_t &= F_1(K^*_t, n_t) \\
w^*_t &= F_2(K^*_t, n_t)
\end{align*}
\]

The planner’s problem first order conditions yield

\[
\begin{align*}
\mu \cdot \beta^t \cdot u'_1(c_{1t}) &= \lambda_t \\
(1-\mu) \cdot \beta^t \cdot u'_2(c_{2t}) &= \lambda_t \\
\lambda_t &= \lambda_{t+1} \cdot F_1(K_{t+1}, n_{t+1})
\end{align*}
\]

Does the solution to these centralized first order conditions also solve the consumers’ decentralized problem? The answer is yes, and we can verify it by using \( p_t = \lambda_t \) to replace in the previous expression for consumer 1 (identical procedure would be valid for consumer 2):

\[
\begin{align*}
\mu \cdot \beta^t \cdot u'_1(c_{1t}) &= p_t \\
\mu \cdot \beta^{t+1} \cdot u'_1(c_{1t+1}) &= p_{t+1}
\end{align*}
\]

So, dividing, we obtain

\[
u'_1(c_{1t}) = \beta_1 \cdot u'_1(c_{1t+1}) \cdot \frac{p_t}{p_{t+1}}
\]
This is the decentralized Euler equation (notice that the multiplier $\mu$ cancels out).

Next we turn to the budget constraint. We have the aggregate expenditure-income equation:

$$
\sum_{t=0}^{\infty} p_t \cdot [c_{1t} + c_{2t} + K_{t+1}] = \sum_{t=0}^{\infty} p_t \cdot [r_t \cdot K_t + w_t \cdot n_t]
$$

By homogeneity of degree 1 of $F(\cdot, \cdot)$, the factor remunerations defined above imply that if the central planner’s resource constraint is satisfied for a $\{c_{1t}, c_{2t}, K_{t+1}\}_{t=0}^{\infty}$ sequence, then this aggregate budget constraint will also be satisfied for that chosen consumption-capital accumulation path.

However, satisfaction of the aggregate budget constraint is not all. We have an additional dilemma: how to split it into two different individual budget constraints. Clearly, we need to split the property of the initial capital between the two agents:

$$
k_{10} + k_{20} = K_0
$$

Does $k_{10}$ contain enough information to solve the dilemma?

First notice that from the central planner’s first order condition

$$
\lambda_t = \lambda_{t+1} \cdot F_1(K_{t+1}, n_{t+1})
$$

we can use the pricing guesses $r_t = F_1(K_t, n_t)$, $p_t = \lambda_t$, and replace to get

$$
p_t = p_{t+1} \cdot r_{t+1}
$$

Therefore we can simplify in the aggregate budget constraint

$$
p_t \cdot K_{t+1} = p_{t+1} \cdot r_{t+1} \cdot K_{t+1}
$$

for all $t$. Then we can rewrite

$$
\sum_{t=0}^{\infty} p_t \cdot [c_{1t} + c_{2t}] = p_0 \cdot r_0 \cdot (k_{10} + k_{20}) + \sum_{t=0}^{\infty} p_t \cdot w_t \cdot n_t
$$

And the individual budgets (where the labor endowment is assigned to each individual) read:

$$
\begin{align*}
\sum_{t=0}^{\infty} p_t \cdot c_{1t} &= p_0 \cdot r_0 \cdot k_{10} + \sum_{t=0}^{\infty} p_t \cdot w_t \cdot n_{1t} \quad \text{(BC1)} \\
\sum_{t=0}^{\infty} p_t \cdot c_{1t} &= p_0 \cdot r_0 \cdot k_{10} + \sum_{t=0}^{\infty} p_t \cdot w_t \cdot n_{1t} \quad \text{(BC2)}
\end{align*}
$$
Notice that none of them include the capital sequence directly, only indirectly via \( w_t \). Recall the central planner’s optimal consumption sequence for Agent 1 \( \{c^*_t\}_{t=0}^{\infty} \) (the one we wish to implement), and the price guesses: \( \{w_t^* = F_2(K^*_t, n_t)\}_{t=0}^{\infty} \) and \( \{p_t^* = \lambda_t^*\}_{t=0}^{\infty} \). Inserting these into (BC1), we have:

\[
\sum_{t=0}^{\infty} p_t^* \cdot c^*_t = p_0^* \cdot r_0^* \cdot k_{10} + \sum_{t=0}^{\infty} p_t^* \cdot w_t^* \cdot n_{1t}
\]

The left hand side \( \sum_{t=0}^{\infty} p_t^* \cdot c^*_t \) is the present market value of planned consumption path for Agent 1. The right hand side is composed by his financial wealth \( p_0^* \cdot r_0^* \cdot k_{10} \) and his "human wealth" endowment \( \sum_{t=0}^{\infty} p_t^* \cdot w_t^* \cdot n_{1t} \). The variable \( k_{10} \) is the adjustment factor that we can manipulate to induce the consumer into the consumption-capital accumulation path that we want to implement.

Therefore, \( k_{10} \) contains enough information: there is a one to one relation between the weight \( \mu \) and the initial capital level (equivalently, the financial wealth) of each consumer. The Pareto optimal allocation characterized by that weight can be implemented with the price guesses defined above, and the appropriate wealth distribution determined by \( k_{10} \). This is the Second Welfare theorem.