1. Consider a growth model with capital accumulation equation $k_{t+1} = f(k_t)$ if $t$ is even and $k_{t+1} = g(k_t)$ if $t$ is odd. Assume that:

   (i) $f(0) = g(0) = 0$.
   (ii) $f'(0)g'(0) > 1$.
   (iii) $\lim_{k \to \infty} f'(g(k))g'(k) < 1$ and $\lim_{k \to \infty} g'(f(k))f'(k) < 1$.
   (iv) $f$ and $g$ are strictly increasing and strictly concave.

Show that, from any initial condition $k_0 > 0$, there is global convergence to a “two-cycle” in which $k_t$ oscillates between two values. How are these values determined?

2. Consider a neoclassical growth model similar to the one that we have discussed in lecture, but in which the level of technology oscillates deterministically between two values $A_H$ and $A_L$, where $A_H > A_L$. In particular, period-$t$ output $y_t$ equals $A_H F(k_t)$ if $t$ is even and equals $A_L F(k_t)$ if $t$ is odd. The planner (“Robinson Crusoe”) seeks to maximize $\sum_{t=0}^{\infty} \beta^t u(c_t)$, given $k_0 > 0$, subject to the resource constraint that $c_t + k_{t+1} = y_t + (1 - \delta)k_t$ and to the nonnegativity constraint $k_{t+1} \geq 0$ for all $t$.

   (a) Formulate the planner’s problem recursively. (Hint: Consider two value functions, one for periods in which the level of technology is high and one for periods in which the level of technology is low. Find a pair of Bellman equations that these functions must satisfy.)

   (b) Let the felicity function $u$ be logarithmic, let $y_t = A_t k_t^\alpha$, and assume that capital depreciates fully in one period (i.e., set $\delta = 1$). Use a guess-and-verify method to find the two value functions in part (a). Describe fully the dynamic behavior of the capital stock.

3. Consider a neoclassical growth model in which the felicity function $u$ has constant elasticity of intertemporal substitution $\sigma^{-1}$:

   $$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma},$$
where $\sigma > 0$ and $u(c) = \log(c)$ if $\sigma = 1$. In addition, assume that $f(k) = Ak^\alpha + (1-\delta)k$, where $A > 0$, $\alpha \in (0,1)$, and $\delta \in [0,1]$. What happens to the speed of convergence to the steady state as $u$ becomes linear (i.e., as $\sigma$ approaches 0)? As $f$ becomes linear (i.e., as $\alpha$ approaches 1)? Try to provide economic intuition for your findings. (Note: As discussed in Section 4.2 in Chapter 4 of the lecture notes, the speed of convergence near the steady state is inversely related to the slope of the decision rule at the steady state.)

4. Consider the planning problem for a basic finite-horizon neoclassical growth model:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^T} \sum_{t=0}^{T} \beta^t \log(c_t),$$

given $k_0 = 10$ and subject to the resource constraint that $c_t + k_{t+1} = Ak_t^\alpha + (1-\delta)k_t$ and to the nonnegativity constraint $k_{t+1} \geq 0$ for all $t \leq T$. Set $\beta = 0.95$, $\delta = 0.1$, and $\alpha = 0.4$. Choose $A$ so that the steady-state value of capital in the corresponding infinite-horizon model is 100.

Solve the model numerically (say, in Matlab) using the “shooting” method described in lecture on October 23: start by guessing a value for $k_1$, solve for $k_2$ from the Euler equation at time 0, then solve for $k_3$ from the Euler equation at time 1, and so on, until $k_{T+1}$ is found ($T$ being the time horizon). Then vary $k_1$ and repeat until the appropriate value of $k_{T+1}$ (what is it?) is found. Find the lowest value for $T$ such that the highest value of capital between periods 0 and $T$ exceeds 90.