HOMEWORK #1

The first two problems on this homework assignment are due on Monday, October 2. The third problem is due on Monday, October 9.

1. Consider a consumer who chooses a (nonnegative) consumption sequence \( \{c_t\}_{t=0}^\infty \) to maximize an objective function subject to the sequence of budget constraints:

\[
c_t + q a_{t+1} = a_t + L, \quad t = 0, 1, 2, \ldots,
\]

where \( c_t \) is consumption in period \( t \), \( a_t \) is asset holdings at the beginning of period \( t \), \( q \) is the (constant) price of a riskfree bond, and \( L > 0 \) is the (constant) amount of labor income in each period.

(a) Suppose that \( 0 < q < 1 \) and that initial asset holdings \( a_0 < L/(q - 1) \). Show that the consumer cannot pay back this debt without violating the nonnegativity constraint on consumption.

(b) Suppose instead that \( q > 1 \). Show that for any initial asset holding \( a_0 < 0 \), the consumer can pay back this debt in finite time without violating the nonnegativity constraint on consumption.

(c) Suppose again that \( q > 1 \). Let \( \{c^*_t\}_{t=0}^\infty \) be a candidate for the optimal consumption sequence. Under the assumption that the consumer’s objective is increasing in consumption, show by construction that the candidate sequence cannot be optimal. (Hint: Show that the consumer can adjust his asset holdings so as to increase consumption in period 0, leaving consumption in all future periods unchanged, without violating either the nonnegativity constraint on consumption or the requirement that debts be repaid.)

This problem shows that the consumer’s problem is not well-defined if \( q > 1 \) (because utility is unbounded) unless one imposes an arbitrary constraint on borrowing.

2. Consider a consumer who seeks to maximize \( \sum_{t=0}^\infty \beta^t u(c_t) \), given \( a_0 \), subject to the same sequence of budget constraints as in the first problem and to the no-Ponzi-game condition \( \lim_{t \to \infty} q^t a_t \geq 0 \) (assume \( q < 1 \)). The discount factor \( \beta \in (0, 1) \). The felicity function \( u \) is strictly increasing and strictly concave.
(a) Suppose that \( q = \beta \). Show that the optimal action for the consumer is to keep his asset holdings constant at \( a_0 \). (Hint: Guess that the optimal decision rule takes the form \( a_{t+1} = \gamma_0 + \gamma_1 a_t \) for all \( t \) and then show that \( \gamma_0 = 0 \) and \( \gamma_1 = 1 \) are the unique choices that satisfy both the Euler equation and the no-Ponzi-game condition.)

(b) Suppose instead that \( q > 1 \) and that the no-Ponzi-game condition is replaced by the constraint that \( a_{t+1} \geq 0 \) for all \( t = 0, 1, 2, \ldots \). (Assume too that initial asset holdings \( a_0 \geq 0 \).) What is the limiting behavior of the consumer’s optimal asset holdings as \( t \) gets large?

3. In this problem, you will write a computer program to obtain numerical solutions to a model studied by Bewley (1977) in an article entitled “The Permanent Income Hypothesis: A Theoretical Formulation” (Journal of Economic Theory, Vol. 16, pp. 252–292). A consumer seeks to maximize \( E_0 \sum_{t=0}^{\infty} \beta^t \log(c_t) \), given \( w_0 \) and \( a_0 \geq 0 \), subject to the sequence of budget constraints:

\[
c_t + a_{t+1} = a_t + w_t, \quad t = 0, 1, 2, \ldots,
\]

where \( w_t \) is labor income in period \( t \). Assume that the stochastic process for labor income is a two-state Markov chain with \( w_t \in \{w_L, w_H\} \) and \( P(w_{t+1} = w_j|w_t = w_i) = \pi_{ij} \) for \( i, j = L, H \). (In this problem, you can think of assets as money, which has a net rate of return of zero and which cannot be held in negative amounts.)

Calibrate the model as follows: let \( \beta = 0.99 \), \( w_L = 0 \) (unemployment), \( w_H = 1 \) (employment at a wage equal to 1), \( \pi_{LH} = d^{-1} \), and

\[
\pi_{HH} = 1 - \frac{u}{d(1-u)},
\]

where the unemployment rate (the fraction of the time that the consumer is unemployed) \( u = 0.1 \) and the average duration of an unemployment spell \( d = 2 \).

(a) Verify that for the specified settings for the \( \pi_{ij} \)'s the unemployment rate (or fraction of the time that the consumer is unemployed) is indeed \( u \).

(b) Use value function iteration on a discrete grid to compute an approximation to the optimal decision rule for savings. As a preliminary step, write the Bellman equation for this problem, letting \( v(a, w) \) be the value function and \( a' \) be the choice variable in the current period. Then proceed as follows:

i. Restrict asset holdings to lie on an equally-spaced grid of \( n \) points on the interval \([0.01, 10]\); call this set of grid points \( A \). The state space \( S \) is then \( A \times \{w_L, w_H\} \).
ii. Choose a set of initial values $v_{ij}^0, i = 1, \ldots, n, j = L, H$, for the value function on $S$. (For example, set $v_{ij}^0 = 0$ for all $i$ and $j$.)

iii. For each of the pairs $(a_i, w_j) \in S$, find the value of $a' \in A$ that maximizes the right-hand side of the Bellman equation, assuming that the $v_{ij}^0$’s determine the value function on $S$. Evaluate the right-hand side of the Bellman equation at the optimal value of $a'$ to obtain a new set of values $v_{ij}^1, i = 1, \ldots, n, j = L, H$.

iv. Set $v_{ij}^0 = v_{ij}^1$ for all $i$ and $j$ and repeat the second, third, and fourth steps until $\max_{ij} |v_{ij}^1 - v_{ij}^0| < \epsilon$, where $\epsilon$ is “close” to 0 (say, $10^{-6}$).

Submit graphs of both the approximate value function and the approximate optimal decision rule. (Set the number of points $n = 100$.)

(c) Use the approximate decision rule and a pseudo random number generator to generate a long time series (say, $10^5$ observations) for the consumer’s asset holdings. Use this time series to calculate an estimate of the consumer’s average asset holdings during his (infinite) lifetime.

(d) Repeat parts (b) and (c) for $\beta = 0.9$ and $\beta = 0.95$. Give an intuitive explanation for your findings.

(e) What is the limit of the consumer’s average asset holdings as $\beta$ approaches 1?