HOMEWORK #4

1. Consider the Bellman equation for the deterministic growth model:

\[ v(k) = \max_{k'} \left[ u(f(k) - k') + \beta v(k') \right], \]

where \( f(k) = Ak^\alpha + (1 - \delta)k \) and \( u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \) (\( u(c) = \log(c) \) when \( \sigma = 1 \)). Pick a set of parameter values and then find a numerical approximation to the value function \( v \) using four approaches:

(a) Approximate \( u(f(k) - k') \) by a second-order Taylor series at the steady-state value of \( k \) and \( k' \) and then iterate on the matrix Ricatti equation to find the coefficients of the quadratic (approximation to the) value function.

(b) Restrict capital to lie on a finite set of grid points and then iterate on the Bellman equation to determine the value function on the grid points. Use Howard policy improvement to increase the rate of convergence.

(c) Approximate the value function by a cubic spline defined on a set of grid points for capital, but do not restrict optimal choices for capital to lie on the grid. Iterate on the Bellman equation (using Howard policy improvement) to determine the value function on the grid points (and use cubic spline interpolation to evaluate the value function at points not on the grid).

(d) Repeat part (c) but approximate the value function by a Chebyshev polynomial rather than by a cubic spline. The number of grid points should be at least as large as the number of polynomial coefficients. When the number of grid points is equal to the number of coefficients, use Chebyshev interpolation to compute the value function at points not on the grid; otherwise, use the Chebyshev regression algorithm described on p. 223 of Judd’s Numerical Methods in Economics to compute explicitly the coefficients of the approximating Chebyshev polynomial. For the case where the number of grid points exceeds the number of coefficients in the Chebyshev polynomial, iterate on the Bellman equation until the coefficients (rather than the values on the grid points) converge.

Compare both the approximate value functions and the associated approximate decision rules delivered by these four approaches. For the last three approaches investigate
how the approximate value functions and decision rules change as the number of grid points (or polynomial coefficients) increases.

2. Derive the (functional) Euler equation associated with the dynamic programming problem in the first problem. This equation determines the optimal decision rule \( g \). Use the Euler equation to find an approximation to \( g \) using five approaches:

(a) Linearize the Euler equation about the steady state, guess that the decision rule for tomorrow’s capital is affine in current capital, and then choose the (two) coefficients in the decision rule so that the linearized Euler equation is satisfied for all values of the state variable (i.e., current capital).

(b) Use a perturbation method: Differentiate the Euler equation at the steady state \( n \) times and then use these identities to solve for the first \( n \) derivatives of the decision rule at the steady state. Use these derivatives to define an \( n \)th-order Taylor series approximation to the decision rule.

(c) Let the decision rule be a (Chebyshev) polynomial of order \( n \) and then use (Chebyshev) collocation to determine the \( n+1 \) coefficients. That is, choose the coefficients so that the error in the Euler equation is equal to 0 at \( n + 1 \) (judiciously chosen) values of the state variable.

(d) Use the den Haan-Marcet version of parameterized expectations: Let the right-hand side of the Euler equation be a function of current capital, approximate this function by a Chebyshev polynomial, and then use Chebyshev collocation to determine the polynomial coefficients.

(e) Repeat part (d) but use the Wright-Williams version of parameterized expectations in which the right-hand side of the Euler equation is viewed as a function of tomorrow’s state variable.

Compare the approximate decision rules delivered by these five approaches. In addition, compare these decision rules to the ones computed in the first problem. For the last four approaches investigate how the approximate decision rules change as the order of the approximating polynomial increases.