1. (a) To start, write down the explicit problem facing the household. It is

\[
\max_{c_0, l_0, c_1, L^m, L^h, L^o, s} u(c_0, l_0, c_1) \tag{1}
\]

subject to

\[
c_0 \leq y_0 + w(L^m - L^h) - s \tag{2}
\]

\[
p_1 c_1 \leq (1 + r)s + f(A_0, L) \tag{3}
\]

\[
L^E = L^m + L^o + l_0 \tag{4}
\]

\[
L = L^h + L^o \tag{5}
\]

and non-negativity constraints. $L^m, L^h,$ and $L^o$ are own labor sold on market, hired labor, and own labor used on the farm, respectively. $c_0$ is the numeraire. The endowment of labor is $L^E$. Repeated substitution of constraints 2-4 into constraint 1 gets you to the full income budget constraint:

\[
c_0 + \frac{1}{1 + r} p_1 c_1 + w l_0 \leq y_0 + w L^E + \frac{1}{1 + r} \Pi \tag{6}
\]

where $\Pi \equiv A_0 \tilde{f}(\frac{L}{A_0}) - w(1 + r)L$ (I’ve obviously made use of CRTS of $f$).

Since $L$ appears only on the RHS of (6), nonsatiation implies that at any solution, $\Pi = \Pi^* \equiv \max_L A_0 \tilde{f}(\frac{L}{A_0}) - w(1 + r)L$. Hence $L^* \equiv \arg \max_L A_0 \tilde{f}(\frac{L}{A_0}) - w(1 + r)L$ satisfies

\[
\tilde{f}(\frac{L^*}{A_0}) = w(1 + r) \tag{7}
\]

which is unique by the strict concavity of $\tilde{f}$. Hence for any $A$, $\frac{L^*}{A}$ is a constant defined by $\tilde{f}^{-1}[w(1 + r)]$.

(b) We can do this in a number of ways. I’ll take what I think is a reasonably simple approach, but not the simplest (that would be to simply set $s = 0$ above). Add
to the above set of constraints the liquidity constraint

$$s \geq 0.$$  

(8)

Obviously, if $s^* \geq 0$ in the above problem, then nothing changes and we maintain the result that $\frac{L^*}{A}$ is constant as $A$ varies. However, if $s^* < 0$, this may no longer be true. The simplest way of showing this is to start with any $A_0$ such that the solution to (1)-(5) above such that $s^* > 0$ and $L_0^* = A_0 * \tilde{f}^{r-1} [w(1 + r)]$. Now consider some $A_{big}$ such that $L_{big} = A_{big} * \tilde{f}^{r-1} [w(1 + r)] > y_0 / w + L^E$. The amount of labor needed to farm $A_{big}$ with the same intensity as $A_0$ is so large that the household cannot afford to hire it, even if it consumes nothing. Hence the intensity of cultivation must fall with $A$ for sufficiently large $A$.

2. Let $s$ index (discrete) states, each of which is a realization of $(y_1, y_2)$ (it’s just as easy, and almost identical, to do this with a continuous state space). So the probability of $s$ is $f_1 f_2 \equiv \pi^s$. A pareto efficient allocation within the household must satisfy

$$\max_{c_1^s, c_2^s} \sum_s \pi^s [u_1(c_1^s) + \lambda u_2(c_2^s)]$$

subject to

$$c_1^s + c_2^s = y_1^s + y_2^s$$

(9)

for some constant $\lambda$. Taking foc wrt to $c_2^s$ and $c_1^s$ dividing gives us

$$\frac{\lambda \pi^s u'_2(c_2^s)}{\pi^s u'_1(c_1^s)} = 1,$$

(10)

where $\pi_s$ is the lagrange multiplier on the resource constraint in state $s$. Thus we have the familiar

$$\frac{u'_2(c_2^s)}{u'_1(c_1^s)} = \frac{1}{\lambda}$$

(11)

for all $s$.

(a) (12) and (10) together define a unique $c_i^s$ as a function of $\lambda$ and $k^s \equiv y_1^s + y_2^s$. 

2
(b) For CARA preferences, \( u_i'(c_i) = \sigma_i e^{-\sigma_i c_i} \). Substitute these into (12) to find

\[
\frac{\sigma_2 e^{-\sigma_2 (y_1 + y_2 - \rho)}}{\sigma_1 e^{-\sigma_1 \rho}} = \frac{1}{\lambda} \tag{13}
\]

taking logs

\[
\ln \sigma_2 - \ln \sigma_1 - \sigma_2 (y_1 + y_2 - \rho) + \sigma_1 \rho = -\ln(\lambda)
\]

or

\[
\rho = \frac{\sigma_2}{\sigma_1 + \sigma_2} (y_1 + y_2) + \frac{1}{\sigma_1 + \sigma_2} (\ln \sigma_1 - \ln \sigma_2 - \ln(\lambda)).
\]

Note that the slope of the sharing rule is decreasing in the risk aversion of person 1 relative to person 2, and of course the amount going to 1 is increasing in the pareto weight of person 1 relative to person 2 (remember, \( \lambda \) is 2’s weight relative to 1).

3. See ps2.do