1 Gains from Trade

1.1 Bilateral Trading

Suppose that a seller values an object at \( c \), and a potential buyer values the same object at \( v \). Trade could occur at a price \( p \), in which case the payoff to the seller is \( p - c \) and to the buyer is \( v - p \). We assume for now that there is only one buyer and one seller, and only one object that can potentially be traded.

Whenever \( v > c \) there is the possibility for a mutually beneficial trade at some price \( c \leq p \leq v \). Any such allocation results in both players receiving non-negative returns from trading and so both are willing to participate (\( p - c \) and \( v - p \) are non-negative).

There are many prices at which trade is possible. And each of these allocations, consisting of whether the buyer gets the object and the price paid, is efficient in the following sense:

**Definition 1** An allocation is pareto efficient if there is no other allocation that makes at least one agent strictly better off, without making any other agent worse off.

1.2 Experimental Evidence

This framework can be extended to consider many buyers and sellers, and to allow for production. One of the most striking examples comes from international trade. We are interested, not only in how specific markets function, but also in how markets should be organized or designed.

There are many examples of markets, such as the NYSE, Nasdaq, E-Bay and Google. The last two consist of markets that were recently created where they did not exist before. So we want to consider not just existing markets, but also the creation of new markets.
Before elaborating on the theory, we will consider three experiments that illustrate how these markets function. We can then interpret the results in relation to the theory. Two types of cards (red and black) with numbers between 2 and 10 are handed out to the students. If the student receives a red card they are a seller, and the number reflects their cost. If the student receives a black card they are a buyer, and this reflects their valuation. The number on the card is private information. Trade then takes place according to the following three protocols.

1. Bilateral Trading: One seller and one buyer are matched before receiving their cards. The buyer and seller can only trade with the individual they are matched with. They have 5 minutes to make offers and counter offers and then agree (or not) on the price.

2. Pit Market: Buyer and seller cards are handed out to all students at the beginning. Buyers and sellers then have 5 minutes to find someone to trade with and agree on the price to trade.

3. Double Auction: Buyer and seller cards are handed out to all students at the beginning. The initial price is set at 6 (the middle valuation). All buyers and sellers who are willing to trade at this price can trade. If there is a surplus of sellers the price is decreased, and if there is a surplus of buyers then the price is increased. This continues for 5 minutes until there are no more trades taking place.

The outcomes of these experiments are interpreted in the first problem set.

1.3 Multiple Buyers and Multiple Sellers

Suppose now that there are N potential buyers and M potential sellers. We can order their valuations and costs by

\[ v_1 > v_2 > ... > v_N \]
\[ c_1 < c_2 < .... < c_M \]

We still maintain the assumption that buyers and sellers only demand or supply at most one object each. It is possible that \( N \neq M \), but we assume that the number of potential traders is finite.

It is possible to realize the efficient trades through a simple procedure. We match the remaining highest valuation buyer with the lowest cost remaining seller until trade is no longer efficient. There will be \( n^* \) trades where \( v_{n^*} \geq c_{n^*} \) and \( v_{n^* + 1} < c_{n^* + 1} \).

FIGURE 1

The value generated by the market when \( n \) people trade is aggregate value of those who buy minus the costs of producing. The gross surplus
of the buyers is given by
\[ V(n) = \sum_{k=1}^{n} v_k \]
and the gross cost of producing is
\[ C(n) = \sum_{k=1}^{n} c_k \]

The efficient number of trades \( n^* = \arg \max_n \{ V(n) - C(n) \} \). The marginal decision then is to find \( n^* \) such that
\[
V(n^*) - V(n^* - 1) \geq C(n^*) - C(n^* - 1) \\
V(n^* + 1) - V(n^*) < C(n^* + 1) - C(n^*)
\]

which is, of course, equivalent to
\[
v_{n^*} \geq c_{n^*} \\
v_{n^*+1} < c_{n^*+1}
\]

What about price? If \( p^* \in [c_{n^*}, v_{n^*}] \) then the market will exhaust all efficient trading opportunities. If there is a market maker who sets price \( p^* \) then we have that for all \( n \leq n^* \)
\[
v_n - p^* \geq 0 \\
p^* - c_n \geq 0
\]
and for all \( n > n^* \)
\[
v_n - p^* < 0 \\
p^* - c_n < 0
\]

So it is possible to achieve the efficient outcome with a single price. In this way we can use prices to decentralize trade.

Some of the matches that make sense in a bilateral trading market will not make sense in a pit market. Consider, for example, the match of \( v_1 \) and \( c_{n^*+1} \). Since \( v_1 - c_{n^*+1} > 0 \), if that buyer and seller were alone together on an island, it would make sense for them to trade. However, once that island is integrated with the rest of the economy (perhaps a ferry service allows the traders on this island to reach other islands), that pair would break down in favor of a more efficient trade. Specifically, the buyer would be able to find someone to sell them the object at a price lower then \( c_{n^*+1} \).
There are several important issues associated with the description of pit market.

- How are prices determined?
- The buyers and sellers can only buy/provide one unit of the good.
- There is only one type of good for sale, and all units are identical.
- There is complete information about the going price and also the valuations (the seller does not know anything that would affect the buyer’s valuation).

Above, we considered the case where there were a finite number of buyers and sellers. We could instead consider the case where there are a continuum of buyers and sellers, and replace consideration of the number of trades with the fraction of buyers who purchase the object. The marginal decision then goes from considering \( n - (n - 1) \) to considering \( dn \): a small change in the proportion buyers who buy. We can then write the aggregate valuations and costs as

\[
V(n) = \int_0^n v_k dk
\]

\[
C(n) = \int_0^n c_k dk
\]

and the optimality condition becomes

\[
\frac{dV(n^*)}{dn} = \frac{dC(n^*)}{dn}
\]

which is, of course, true if and only if \( v_{n^*} = c_{n^*} \). This is then a "serious marginal condition" that the marginal buyer and marginal seller must have the same valuation. This, in turn, uniquely ties down the price as \( p^* = v_{n^*} = c_{n^*} \). We can then consider "life at the margin" and see what the effects of small changes are on the economy.

2 Choice

In the decision problem in the previous section, the agents had a binary decision: whether to buy(sell) the object. However, there are usually more than two alternatives. The price at which trade could occur, for example, could take on a continuum of values. In this section we will look more closely at preferences, and determine when it is possible to represent preferences by "something handy", which is a utility function.
Suppose there is a set of alternatives \( X = \{x_1, x_2, \ldots, x_n\} \) for some individual decision maker. We are going to assume, in a manner made precise below, that two features of preferences are true.

a.) There is a complete ranking of alternatives.

b.) "Framing" does not affect decisions.

We refer to \( X \) as a choice set consisting of \( n \) alternatives, and each alternative \( x \in X \) is a consumption bundle of \( k \) different items. For example, the first element of the bundle could be food, the second element could be shelter and so on. We will denote preferences by \( \succ \), where \( x \succ y \) means that "\( x \) is strictly preferred to \( y \)". All this means is that when a decision maker is asked to choose between \( x \) and \( y \) they will choose \( x \). Similarly, \( x \succeq y \), means that "\( x \) is weakly preferred to \( y \)" and \( x \sim y \) indicates that the decision maker is "indifferent between \( x \) and \( y \)". We make the following three assumptions about preferences.

**Axiom 2 Completeness.** For all \( x, y \in X \) either \( x \succ y, y \succ x, \) or \( y \sim x \).

This first axiom simply says that, given two alternatives the decision maker can compare the alternatives, and will strictly prefer one of the alternatives, or will be indifferent.

**Axiom 3 Transitivity.** For all triples \( x, y, z \in X \) if \( x \succ y, \) and \( y \succ z \) then \( x \succ z \).

Very simply, this axiom imposes some level of consistency on choices. For example, suppose there were three potential travel locations, Tokyo(T), Beijing(B), and Seoul(S). If a decision maker, when offered the choice between Tokyo and Beijing chose to go to Tokyo, and when given the choice between Beijing and Seoul choose to go to Beijing, then this axiom simply says that if they were offered a choice between a trip to Tokyo or a trip to Seoul, they would choose Tokyo. This is because they have already demonstrated that they prefer Tokyo to Beijing, and Beijing to Seoul, so preferring Seoul to Tokyo would mean that their preferences are inconsistent.

**Axiom 4 Reflexivity.** For all \( x \in X, x \succeq x \) (equivalently, \( x \sim x \)).

The final axiom is made for technical reasons, and simply says that a bundle cannot be strictly preferred to itself. Such preferences would not make sense.

These three axioms allow for bundles to be ordered in terms of preference. In fact, these three conditions are sufficient to allow preferences to be represented by a utility function.
Before elaborating on this, we consider an example. Suppose there are two goods, Wine and Cheese. Suppose there are four consumption bundles \( z = (2, 2), y = (1, 1), a = (2, 1), b = (1, 2) \) where the two elements of the vector represent the amount of wine or cheese. Most likely, \( z \succ y \) since it provides more of everything (i.e. wine and cheese are "goods"). It is not clear how to compare \( a \) and \( b \). What we can do is consider which bundles are indifferent with \( b \). This is an indifference curve. We can define it as

\[
I_b = \{ x \in X | b \sim x \}.
\]

FIGURE 2

We can then (if we assume that more is better) compare \( a \) and \( b \) by considering which side of the indifference curve \( a \) lies on: bundles above and to the right are more preferred, bundles below and to the left are less preferred. This reduces the dimensionality of the problem. We can speak of the "better than \( b \)" set as the set of points weakly preferred to \( b \). These preferences are "ordinal": we can ask whether \( x \) is in the better than set, but this does not tell us how much \( x \) is preferred to \( b \).

Suppose I want to increase my consumption of good 1 without changing my level of well-being. The amount I must change \( x_2 \) to keep utility constant, \( \frac{dx_2}{dx_1} \) is the marginal rate of substitution. Most of the time we believe that individuals like moderation. This desire for moderation is reflected in convex preferences.

**Definition 5** A preference relation is convex if for all \( y \) and \( y' \) with \( y \sim y' \) and all \( \alpha \in [0, 1] \) we have that \( \alpha y + (1 - \alpha)y' \succeq y \sim y' \).

While convex preferences are usually assumed, there could be instances where preferences are not convex. For example, there could be returns to scale for some good.

Examples: perfect substitutes, perfect compliments. Both of these preferences are convex.

FIGURE 3

Notice that indifference curves cannot intersect. If they did we could take two points \( x \) and \( y \), both to the right of the indifference curve the other lies on. We would then have \( x \succ y \succ x \), but then by transitivity \( x \succ x \) which contradicts reflexivity. So every bundle is associated with one, and only one, welfare level.

3 Utility Functions

What we want to consider now is whether we can take preferences and map them to some sort of utility index. If we can somehow represent
preferences by such a function we can apply mathematical techniques to make the consumer’s problem more tractable. Working with preferences directly requires comparing each of a possibly infinite number of choices to determine which one is most preferred. Maximizing an associated utility function is often just a simple application of calculus. If we take a consumption bundle \( x \in \mathbb{R}^N_+ \) we can take a utility function as a mapping from \( \mathbb{R}^N_+ \) into \( \mathbb{R} \).

**Definition 6** A utility function (index) represents a preference profile \( \succeq \) if for all \( x, y \in \mathbb{R}^N_+ \), \( x \succeq y \) if and only if \( u(x) \geq u(y) \).

Is it always possible to find such a function? The following result, due to Debreu, shows that such a function exists under the three assumptions about preferences we made above.

**Proposition 7** Every (continuous) preference ranking can be represented by a (continuous) utility function.

This result can be extended to environments with uncertainty, as was shown by Savage. Consequently, we can say that individuals behave as if they are maximizing utility functions, which allows for marginal and calculus arguments. There is, however, one qualification. The utility function that represents the preferences is not unique.

**Remark 8** If \( u \) represents preferences, then for any increasing function \( f : \mathbb{R}^+ \to \mathbb{R} \), \( f(u(x)) \) also represents the same preference ranking.

In the previous section, we claimed that preferences usually reflect the idea that "more is better", or that preferences are monotone.

**Definition 9** The utility function (preferences) are monotone increasing if \( x \succeq y \) implies that \( u(x) \geq u(y) \) and \( x \succ y \) implies that \( u(x) > u(y) \).

One feature that monotone preferences rule out is (local) satiation, where one point is preferred to all other points nearby. For economics the relevant decision is maximizing utility subject to limited resources. The leads us to consider constrained optimization.

4 Constrained Optimization

Consumers are typically endowed with money \( m \), which determines which consumption bundles are affordable. The budget set consists of all consumption bundles such that \( p \cdot x = \sum_{k=1}^{n} p_k x_k \leq m \). The consumer’s
problem is then to find the point on the highest indifference curve that is in the budget set. At this point the indifference curve must be tangent to the budget line, \( \frac{dx_2}{dx_1} = \frac{-p_1}{p_2} \) which defines how much \( x_2 \) must decrease by if the amount of consumption of good 1 is increased by \( \Delta x_1 \) or \( dx_1 \) for the bundle to still be affordable. It reflects the opportunity cost, as money spent on good 1 cannot be used to purchase good 2.

**FIGURE 4**

The marginal rate of substitution reflects the relative benefit from consuming different goods. We can define it as

\[
MRS = \frac{MU_1}{MU_2} = \frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}},
\]

where \( MU_i \) refers to the marginal utility of consuming good \( i \). The slope of the indifference curve is \(-MRS\) so the relevant optimality condition, where the slope of the indifference curve equals the slope of the budget line, is

\[
\frac{p_1}{p_2} = \frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}}.
\]

We could equivalently talk about equating marginal utility per dollar. If

\[
\frac{\frac{\partial u(x)}{\partial x_2}}{p_2} > \frac{\frac{\partial u(x)}{\partial x_1}}{p_1}
\]

then one dollar spent on good 2 generates more utility then one dollar spent on good 1. So shifting consumption from good 1 to good 2 would result in higher utility. So, to be at an optimum we must have the marginal utility per dollar equated across goods.

Does this mean then that we must have \( \frac{\partial u(x)}{\partial x_i} = p_i \) at the optimum? No. Such a condition wouldn’t make sense since we could rescale the utility function. We could instead rescale the equation by a factor \( \lambda \geq 0 \) that converts "money" into "utility". We could then write \( \frac{\partial u(x)}{\partial x_i} = \lambda p_i \). Here, \( \lambda \) reflects the marginal utility of money.

**Example 10** Consider now an example of quasi-linear utility in one good. Suppose there is some benefit to holding money (perhaps there will be opportunities for consumption in the future). Then we write

\[
V(x, m) = u(x) + m - px.
\]

the money term enters linearly, hence the name. The optimal condition gives

\[
u'(x) - p = 0
\]

where
so the marginal benefit (utility from consuming $x$) equals the marginal cost (foregone money). In this example, $\frac{\partial V(x,m)}{\partial m} = 1$ is the marginal utility of money.

The consumer’s problem is to maximize utility subject to a budget constraint. There are two ways to approach this problem. The first approach involves writing the last good as a function of the previous goods, and then proceeding with an unconstrained maximization. Consider the two good case. The budget set consists of the constraint that $p_1 x_1 + p_2 x_2 \leq m$. So the problem is

$$\max u(x) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq m$$

But notice that whenever $u$ is (locally) non-satiated then the budget constraint holds with equality since there in no reason to hold money that could have been used for additional valued consumption. So, $p_1 x_1 + p_2 x_2 = m$, and so we can solve for $x_2 = \frac{m - p_1 x_1}{p_2}$. Now we can maximize $u(x_1, \frac{m - p_1 x_1}{p_2})$ as the standard single variable maximization problem.

**Example 11** Consider the Cobb-Douglas Utility Function

$$u(x) = x_1^\alpha x_2^{1 - \alpha}$$

then substituting the expression for $x_2$ gives

$$u(x_1) = x_1^\alpha \left( \frac{m - p_1 x_1}{p_2} \right)^{1 - \alpha}$$

Taking the derivative

$$u'(x_1) = \alpha x_1^{\alpha - 1} \left( \frac{m - p_1 x_1}{p_2} \right)^{1 - \alpha} \left( \frac{m - p_1 x_1}{p_2} \right)^{-\alpha} \left( \frac{-p_1}{p_2} \right)$$

$$= x_1^{\alpha - 1} \left( \frac{m - p_1 x_1}{p_2} \right)^{-\alpha} \left[ \alpha \left( \frac{m - p_1 x_1}{p_2} \right) + (1 - \alpha) x_1 \left( \frac{-p_1}{p_2} \right) \right]$$

so the FOC is satisfied when

$$\alpha (m - p_1 x_1) - (1 - \alpha) x_1 p_1 = 0$$

which holds when

$$x_1 = \frac{\alpha m}{p_1}.$$ 

And so

$$x_2 = \frac{m - p_1 x_1}{p_2} = \frac{(1 - \alpha) m}{p_2}.$$
Several important features of this example are worth noting. First of all, $x_1$ does not depend on $p_2$. Also, the share of income spent on each good $\frac{m p_i}{m}$ does not depend on price or wealth. What is going on here? When the price of one good, $p_2$, increases there are two effects. First, the price increase makes good 1 relatively cheaper ($\frac{p_1}{p_2}$ decreases). This will cause consumers to "substitute" toward the relatively cheaper good. There is also another effect. When the price increases the individual becomes poorer in real terms, as the set of affordable consumption bundles becomes strictly smaller. The Cobb-Douglas utility function is a special case where this "income effect" exactly cancels out the substitution effect, so the consumption of one good is independent of the price of the other goods.

While this approach is simple enough, there are situations where it will be difficult to apply. The procedure requires that we know, before the calculation, that the budget constraint actually binds. In many situations there may be other constraints (such as a non-negativity constraint on the consumption of each good) and we may not know whether they bind before demands are calculated. Consequently, we will consider a more general approach of Lagrange Multipliers. We want to consider the problem of

$$\max u(x) \text{ s.t. } p \cdot x \leq m$$

The basic idea is as follows. Just as a necessary condition for a maximum in a one variable maximization problem is that the derivative equals 0 ($f'(x) = 0$), a necessary condition for a maximum in multiple variables is that all partial derivatives are equal to 0 ($\frac{\partial f(x)}{\partial x_i} = 0$). To see why, recall that the partial derivative reflects the change as $x_i$ increases and the other variables are all held constant. If any partial derivative was positive, then holding all other variables constant while increasing $x_i$ will increase the objective function (similarly, if the partial derivative is negative we could decrease $x_i$). We also need to ensure that the solution is in the budget set, which typically won't happen if we just try to maximize $u$. Basically, we impose a "cost" on consumption, proceed with unconstrained maximization for the induced problem, and set this cost so that the maximum lies in the budget set.

Recall the quasi-linear (one-good) problem from the previous section. In this problem the goal is to maximize $u(x) + [m - px]$ where the $m - px$ term provides a cost of spending money on consumption, providing FOC $u'(x) = p$. We have seen that this corresponds to the case where the marginal utility of money is $\lambda = 1$. One unusual thing about such an objective is that the chosen level of consumption does not depend on
the amount of money. So it is possible that the chosen consumption level has, $px > m$, even though such a bundle is unaffordable. It is also possible that $px < m$, so the individual won’t spend all of their money even when utility functions are locally non-satiated. However, we can choose a $\lambda$ where $u'(x) = \lambda p$ and $m = px$ whenever the utility function is locally non-satiated.

We consider the problem of maximizing with respect to $x$ the Lagrangian

$$L(x) = u(x) + \lambda[m - p \cdot x]$$

where $\lambda \geq 0$ and either $p \cdot x = m$ or $\lambda = 0. \lambda = 0$ corresponds to the case where the constraint is not binding.

The solution to this unconstrained problem is the same as the solution to the constrained maximization problem. Maximizing the Lagrangian results in the FOCs

$$\frac{\partial L(x)}{\partial x_i} = \frac{\partial u(x)}{\partial x_i} - \lambda p_i = 0 \text{ for all } i = 1, \ldots, N$$

$$\frac{\partial L(x)}{\partial \lambda} = m - p \cdot x = 0 \text{ or } \lambda = 0$$

The final condition could be re-written as

$$\lambda[m - p \cdot x] = 0.$$

Notice that these FOCs imply that

$$\frac{\frac{\partial u(x)}{\partial x_1}}{p_1} = \lambda = \frac{\frac{\partial u(x)}{\partial x_2}}{p_2},$$

which is precisely the MRS=price ratio condition for optimality that we saw before.

Finally, it should be noted that the FOCs are necessary for optimality, but they are not, in general, sufficient for the solution to be a maximum. However, whenever $u(x)$ is a concave function the FOCs are also sufficient to ensure that the solution is a maximum. In most situations, the utility function will be concave.

Example 12 We can consider the problem of deriving demands for a Cobb-Douglas utility function using the Lagrangian approach. The associated Lagrangian is

$$L(x) = x_1^\alpha x_2^{1-\alpha} + \lambda[m - p_1 x_1 - p_2 x_2]$$
which yields the associated FOCs

\[
\frac{\partial L(x)}{\partial x_1} = \alpha x_1^{\alpha - 1} x_2^{1-\alpha} - \lambda p_1 = \alpha \left( \frac{x_2}{x_1} \right)^{1-\alpha} - \lambda p_1 = 0
\]

\[
\frac{\partial L(x)}{\partial x_2} = (1 - \alpha) x_1^\alpha x_2^{-\alpha} - \lambda p_2 = (1 - \alpha) \left( \frac{x_1}{x_2} \right)^\alpha - \lambda p_2 = 0
\]

\[\lambda [m - p_1 x_1 - p_2 x_2] = 0\]

Notice first that since it is not possible that \(x_2 x_1\) and \(x_1 x_2\) are both 0 we cannot have a solution to the first two equations with \(\lambda = 0\). Consequently we must have that \(p_1 x_1 + p_2 x_2 = m\). Solving for \(\lambda\) in the above equations tells us that

\[\lambda = \frac{\alpha (x_2/x_1)^{1-\alpha}}{p_1} = \frac{1 - \alpha}{p_2} \frac{(x_1/x_2)^\alpha}{p_1}\]

and so

\[p_2 x_2 = \frac{1 - \alpha}{\alpha} p_1 x_1.\]

Combining with the budget constraint this gives

\[\frac{1 - \alpha}{\alpha} p_1 x_1 + p_1 x_1 = \frac{1}{\alpha} p_1 x_1 = m\]

So

\[x_1 = \frac{\alpha m}{p_1}\]

and

\[x_2 = \frac{(1 - \alpha)m}{p_2}.\]

So we see that the result of the Lagrangian approach is the same as from the first approach.

When we take a monotone transformation of a utility function the underlying preferences represented are not changed. Consequently the consumer’s demands are unaffected, by such a transformation.

Example 13 Notice that

\[\log x_1^\alpha x_2^{1-\alpha} = \alpha \log x_1 + (1 - \alpha) \log x_2\]

so we could write the utility function for Cobb-Douglas preferences by \(u(x) = \alpha \log x_1 + (1 - \alpha) \log x_2\). The associated Lagrangian is

\[L(x) = \alpha \log x_1 + (1 - \alpha) \log x_2 + \lambda [m - p_1 x_1 - p_2 x_2]\]
with associated FOCs

\[
\frac{\partial L(x)}{\partial x_1} = \frac{\alpha}{x_1} - \lambda p_1 = 0
\]

\[
\frac{\partial L(x)}{\partial x_2} = \frac{1 - \alpha}{x_2} - \lambda p_2 = 0
\]

\[
\lambda[m - p_1x_1 - p_2x_2] = 0
\]

Since \( \frac{\alpha}{x_1} > 0 \) we must have \( \lambda > 0 \) so the budget constraint must hold with equality. Solving for \( \lambda \) gives

\[
\lambda = \frac{\alpha}{p_1x_1} = \frac{1 - \alpha}{p_2x_2}
\]

so

\[
p_2x_2 = \frac{1 - \alpha}{\alpha}p_1x_1.
\]

and

\[
\frac{1 - \alpha}{\alpha}p_1x_1 + p_1x_1 = \frac{1}{\alpha}p_1x_1 = m.
\]

So, we can conclude, as in the previous example that

\[
x_1 = \frac{\alpha m}{p_1}
\]

and

\[
x_2 = \frac{(1 - \alpha)m}{p_2}.
\]

5 Corner Solutions

Consider the utility function \( u(x) = \ln x_1 + x_2 \), where \( p_1, p_2 \) and \( m \) are all strictly greater than 0. The first thing to notice is that while the marginal utility of good 2 is constant \( \frac{\partial u(x)}{\partial x_2} = 1 \), the marginal utility of good 1, \( \frac{\partial u(x)}{\partial x_1} = \frac{1}{x_1} \) is a decreasing function of \( x_1 \). The second derivative is negative, \( \frac{\partial^2 u(x)}{\partial x_1^2} = -\frac{1}{x_1^2} < 0 \) because \( \ln \) is a concave function. That is, there is diminishing marginal utility from consuming good 1. Suppose that we consider the problem of maximizing \( u(x) = \ln x_1 + x_2 \) subject to the budget constraint, \( p_1x_1 + p_2x_2 \leq m \). We would expect that the size of the budget would greatly affect the shape of the demands. The consumer will spend everything on good 1 as long as

\[
\frac{MU_1}{p_1} \geq \frac{MU_2}{p_2}
\]
once $x_1$ is large enough that
\[
\frac{MU_1}{p_1} = \frac{MU_2}{p_2}
\]
we reach a "critical point" and it becomes more profitable to spend the remaining money on good 2. This point occurs when
\[
\frac{1}{x_1} = \frac{1}{p_1}
\]
So $x_1^* = \frac{p_2}{p_1}$. So the first $p_1x_1^*$ dollars will be spent on good 1, and any remaining money will be spent on good 2. The maximum amount to be spent on good 1 is simply $m^* = p_1x_1^* = p_2$. So we can now formulate the demand.

\[
\begin{cases}
  x_1 = \frac{m}{p_1}, & m < m^* = p_2 \\
  x_2 = 0
\end{cases}
\]

\[
\begin{cases}
  x_1 = \frac{p_2}{p_1}, & m \geq m^* = p_2 \\
  x_2 = \frac{m-p_2}{p_1}
\end{cases}
\]

We can then graph the effect of income on consumption.

**FIGURE 5**

In the above example, with a small budget it would be preferable to consume negative amounts of good 2. This negative consumption is called "short selling". In most situations this is not reasonable. What would it mean, for example, to consume $-2$ apples? Consequently, we often need to add additional constraints to the problem: that the consumption of each good must be non-negative. When such constraints bind we have what is known as a "corner solution". We can deal with these non-negativity constraints exactly the same way as the budget constraint: by adding Lagrange multipliers associated with each constraint. Unlike with the budget constraint which will hold with equality whenever utility functions are monotone, it may not be clear before beginning the calculation which non-negativity constraint will bind. Instead of just having the budget constraint
\[
p_1x_1 + p_2x_2 \leq m
\]
we also have the constraints
\[
\begin{align*}
x_1 &\geq 0 \\
x_2 &\geq 0.
\end{align*}
\]

Typically the Lagrange multipliers associated with these constraints are denoted by $\mu_1$ and $\mu_2$. 

14
So the associated Lagrangian is

\[ L(x) = u(x) + \lambda[m - p_1 x_1 - p_2 x_2] + \mu_1 x_1 + \mu_2 x_2 \]

which then yields FOCs

\[
\frac{\partial L(x)}{\partial x_1} = \frac{\partial u(x)}{\partial x_1} - \lambda p_1 + \mu_1 = 0 \\
\frac{\partial L(x)}{\partial x_2} = \frac{\partial u(x)}{\partial x_2} - \lambda p_2 + \mu_2 = 0
\]

and the complimentary slackness conditions

\[
\lambda[m - p_1 x_1 - p_2 x_2] = 0 \\
\mu_1 x_1 = 0 \\
\mu_2 x_2 = 0.
\]

This means we are left with 5 equations in 5 unknowns. In the previous example with \( u(x) = \ln x_1 + x_2 \), the first two conditions become

\[
\frac{1}{x_1} - \lambda p_1 + \mu_1 = 0 \\
1 - \lambda p_2 + \mu_2 = 0
\]

For small \( m \), \( x_1 > 0 \) and \( x_2 = 0 \) so \( \mu_1 = 0 \). But the budget constraint guarantees that

\[ x_1 = \frac{m}{p_1} \]

which combined with the first condition says that \( \lambda = \frac{1}{m} \). So

\[ \mu_2 = \frac{p_2}{m} - 1 \]

This is the shadow price of short-selling. That is, how valuable relaxing the non-negativity constraint on good 2 would be.

6 Understanding Lagrangians

Before we proceed we consider the simplest possible constrained problem, the one-dimensional maximization problem, to develop a feel for how this technique works. Let \( f(x) \) be a function from the real line into itself. The problem is to choose \( x \) to maximize \( f(x) \). In the unconstrained problem, the FOC \( f'(x^*) = 0 \) is a sufficient condition for a local extremum. To ensure that said extremum is in fact a local maximum we must consider the SOC \( f''(x^*) \leq 0 \). What if we are interested in a global max? Then we could replace the above SOC with \( f''(x) \leq 0 \) for
all x. This will guarantee that the function is concave and so any local max must be a global max.

Let \( \hat{x} = \arg \max f(x) \), and consider some \( \bar{x} < \hat{x} \) and suppose instead that we are considering the problem of maximizing \( f(x) \) subject to \( x \leq \bar{x} \). Assume that \( f \) is concave (FIGURE 6). We know that \( f'(\hat{x}) = 0 \) but \( \hat{x} \) is not available in the new problem. Is there an analogous expression for the solution to the constrained problem?

If we consider the constrained problem we know that \( f \) is increasing for all \( x \leq \bar{x} \), so we expect the the solution to be \( x^* = \bar{x} \), so a "corner solution". If we set up the Lagrangian we have that

\[
L(x) = f(x) - \lambda(x - \bar{x})
\]

and require that

\[
\begin{align*}
\lambda(x - \bar{x}) &= 0 \\
\lambda &\geq 0
\end{align*}
\]

The Lagrangian consists of both the objective and the constraint and \( x^* \) solves the constrained maximization problem if and only if it maximizes the Lagrangian. The FOC with respect to \( x \) gives the optimality condition

\[
f'(x^*) = \lambda
\]

and if \( \lambda > 0 \) this means that \( x^* = \bar{x} \) so

\[
f'(\bar{x}) = \lambda
\]

That is, \( \lambda \) reflects the marginal value of relaxing the constraint.

How do we know that we wont have \( x^* > \bar{x} \)? Under the original formulation individuals were simply not allowed to choose a bundle that violates the constraint, but under the Lagrangian approach the maximization is unconstrained. The \( \lambda \) term introduces a "price", that ensures that individuals do not choose to consume more then is feasible. If more was consumed than available then this price would have been set too low. So rather than setting unbreakable constraints, the economics approach is to allow for "prices" that make individuals not want to choose impossible amounts given the resource constraints in the economy. This allows for "decentralizing via prices". This will be a central idea behind competitive equilibrium, which will be introduced soon.

7 Demand

So far we have considered only the decisions of single agents trying to make the optimal decisions given some exogenous parameters and con-
straints. In the next sections we will integrate these individual decision makers with other decision makers and consider when the resulting system is in "equilibrium". First, we will consider the behaviour of demands. For given prices and wealth an individual has demands \((x_1(p_1, p_2, m), x_2(p_1, p_2, m))\). The demand for good 1 can be affected by changes in its own price, \(\frac{\partial x_1}{\partial p_1}\), or by a cross-price effect \(\frac{\partial x_1}{\partial p_2}\).

When the price of good 2 changes there are two effects. First, it makes the prices ratio \(\frac{p_1}{p_2}\) change. This is known as the substitution effect. Second, the change in \(p_2\) causes a change in the consumption bundles that are feasible, so an increase in \(p_2\) makes the consumer poorer, while a decrease makes individuals richer. This effect is known as the income effect. Together these two effects constitute a price effect.

One example where the substitution effect is unimportant is the Leontief utility function. Goods are perfect complements and so there is no substitution. Here, \(u(x) = \min\{ax_1, bx_2\}\) so it will be optimal to set \(ax_1 = bx_2\) whatever the prices of the two goods may be. So \(x_2 = \frac{ax_1}{b}\) and from the budget set \(p_1x_1 + p_2x_2 = (p_1 + \frac{ap_2}{b})x_1 = m\). So

\[
x_1 = \frac{bm}{bp_1 + ap_2} \\
x_2 = \frac{am}{bp_1 + ap_2}
\]

If the price of good 1 were to fall the consumption of both goods will increase, since the consumer is made wealthier (more bundles are now affordable).

**FIGURE 7**

### 7.1 Elasticity

In many situations we are interested in knowing how demand changes when price changes. This information would be extremely important, as one example, to a firm trying to set a price for its products that will maximize its profits. One index often used is price elasticity. What elasticity measures, essentially, is what percentage decrease there will be in the quantity purchased if the price were to increase by 1%. More precisely, we define

**Definition 14** The elasticity of substitution \(\epsilon = -\frac{\frac{\partial x_1}{\partial p_1}}{x_1\frac{p_1}{p}}\).

One case to consider is the "constant elasticity of substitution" demand curve. Here \(x = Ap^\epsilon\) so \(\frac{\frac{\partial x}{\partial p}}{\frac{p}{x}} = \frac{Ap^{\epsilon - 1}}{Ap^{\epsilon - 1}} = \epsilon\), which is a constant independent of the price.
The constant elasticity of demand is also convenient since we can then take logarithms and get

\[ \ln x = \ln A + \epsilon \ln p \]

So we can estimate the elasticity of substitution empirically through a simple linear regression.

8 Competitive Equilibrium

So far we have been concerned with a single agent decision problem. Using the tools developed, we can now consider the more interesting situation where many agents are interacting. One way of modelling the interaction between many agents is Competitive Equilibrium, or General Equilibrium. In a competitive equilibrium framework the agents are "endowed" with some vector of commodities, and then proceed to trade at some prices. These prices must be such that the amount offered for trade equals the amount demanded. This trading results in a new allocation, where all mutually beneficial trade has been exhausted. We will conclude this section with two of the most important results in economics.

**Theorem 15** *(First Welfare Theorem)* Every competitive equilibrium is pareto efficient.

**Theorem 16** *(Second Welfare Theorem)* Every pareto efficient allocation can be decentralized as a competitive equilibrium. That is, every pareto efficient allocation is the equilibrium for some endowments.

We now consider the example of an economy with two people and two goods. The framework is far more general and can be extended without complications to any arbitrary number of agents and goods, but we consider only this case to simplify notation. We assume that there are two people in the economy, Robinson and Friday. The budget set is simply \( p_1 x_1 + p_2 x_2 \leq m \) as usual. However, the amount of money an individual has is simply the value of their endowment. If the initial endowment of good 1 and 2 respectively is \( w = (w_1, w_2) \) then the budget set is simply \( p_1 x_1 + p_2 x_2 \leq p_1 w_1 + p_2 w_2 \). Suppose there are two goods, Bananas and Coconuts, then we can write the endowments for Robinson and Friday as \( w_r = (w_{br}, w_{cr}) \) and \( w_f = (w_{bf}, w_{cf}) \) as the endowments of bananas and coconuts respectively.

**FIGURE 8 Edgeworth Box**

So then the problem for Robinson is to solve

\[
\max u_r(x_{br}, x_{cr}) \; \text{s.t.} \; p_b x_{br} + p_c x_{cr} \leq p_b w_{br} + p_c w_{cr} \quad \text{(R)}
\]
and similarly Friday’s problem is to solve

$$\max u_r(x_{bf}, x_{cf}) \text{ s.t. } p_b x_{bf} + p_c x_{cf} \leq p_b w_{bf} + p_c w_{cf} \quad (F)$$

We can then define a competitive equilibrium. This definition can, of course, be extended to more goods, and more agents.

**Definition 17** A Competitive Equilibrium is a set of prices $p = (p_b, p_c)$ and a set of consumption choices $x_r$ and $x_f$ such that at these prices $x_r$ solves the utility maximization problem in (R) and $x_f$ solves (F), and markets clear so that

$$x_{br} + x_{bf} = w_{br} + w_{bf}$$
$$x_{cr} + x_{cf} = w_{cr} + w_{cf}$$

So competitive equilibrium consists of an individual problem and a social problem: agent optimization and market clearing. Basically, this definition means that in a competitive equilibrium all agents take the prices as given, and choose the optimal bundle at these prices. The prices must be set so the aggregate demand for each good, (the sum of how much each agent demands) is equal to the aggregate supply (the sum of all agents’ endowments). We can denote the total endowment of each good by $w_b = w_{br} + w_{bf}$ and $w_c = w_{cr} + w_{cf}$.

We now consider a simple example, where Friday is endowed with the only (perfectly divisible) banana and Robinson is endowed with the only coconut. That is $w_f = (1, 0)$ and $w_r = (0, 1)$. To keep things simple suppose that both agents have the same utility function

$$u(x) = \alpha \sqrt{x_b} + \sqrt{x_c}$$

and we consider the case where $\alpha > 1$, so there is a preference for bananas over coconuts that both agents share.

We can determine the indifference curves for both Robinson and Friday that the initial endowment lies on. The indifference curves are given by

$$u_f(x_b, x_c) = \alpha \sqrt{x_b} + \sqrt{x_c} = \alpha = u_f(1, 0)$$
$$u_r(x_b, x_c) = \alpha \sqrt{x_b} + \sqrt{x_c} = 1 = u_r(0, 1)$$

All the allocations between these two indifference curves are pareto superior to the initial endowment.

**FIGURE 9**
We can define the net trade for Friday (and similarly for Robinson) by

\[ z_{bf} = x_{bf} - w_{bf} \]

\[ z_{cf} = x_{cf} - w_{cf} \]

Notice that since initially Friday had all the bananas and none of the coconuts that

\[ z_{bf} \leq 0 \]

\[ z_{cf} \geq 0 \]

The first welfare theorem tells us that in a competitive equilibrium trade will take place until we have found a pareto efficient allocation, where all mutually beneficial trades have been exhausted. At this point the indifference curves of the two agents must be tangent.

**FIGURE 10**

There could be many Pareto efficient allocations (e.g. Friday gets everything, Robinson gets everything, etc.) but we can calculate which allocations are pareto optimal. If the indifference curves at an allocation are tangent then the marginal rates of substitution must be equated. In this case, the resulting condition is

\[
\frac{\partial u_f}{\partial x_{bf}} = \frac{\alpha}{2\sqrt{x_{bf}}} = \frac{\alpha}{2\sqrt{x_{cf}}} = \frac{\partial u_r}{\partial x_{br}} = \frac{\partial u_r}{\partial x_{cr}}
\]

which means that

\[
\frac{\sqrt{x_{cf}}}{\sqrt{x_{bf}}} = \frac{\sqrt{x_{cr}}}{\sqrt{x_{br}}}
\]

and, of course since there is a total of one unit of each commodity, for market clearing we must have

\[ x_{cr} = 1 - x_{cf} \]

\[ x_{br} = 1 - x_{bf} \]

so

\[
\frac{\sqrt{x_{cf}}}{\sqrt{x_{bf}}} = \frac{\sqrt{1 - x_{cf}}}{\sqrt{1 - x_{bf}}}
\]

and squaring both sides

\[
\frac{x_{cf}}{x_{bf}} = \frac{1 - x_{cf}}{1 - x_{bf}}
\]
which implies that
\[ x_{cf} - x_{cf}x_{bf} = x_{bf} - x_{cf}x_{bf} \]

and so
\[ x_{cf} = x_{bf} \]
\[ x_{cr} = x_{br} \]

so in this example, the pareto efficient allocations are precisely the 45 degree line in the Edgeworth box.

FIGURE 11
What are the conditions necessary for an equilibrium? First we need the conditions for Friday to be optimizing, which consists of the standard FOCs of the maximization problem
\[
\frac{\partial u_f}{\partial x_{bf}} = \lambda_f p_b \\
\frac{\partial u_f}{\partial x_{cf}} = \lambda_f p_c \\
\]
\[ p_b(x_{bf} - w_{bf}) + p_c(x_{cf} - w_{cf}) = 0 \]

where the last condition is that Friday spends all his income. Similarly, Robinson must be optimizing resulting in similar expressions
\[
\frac{\partial u_r}{\partial x_{br}} = \lambda_r p_b \\
\frac{\partial u_r}{\partial x_{cr}} = \lambda_r p_c \\
\]
\[ p_b(x_{br} - w_{br}) + p_c(x_{cr} - w_{cr}) = 0 \]

and finally we require market clearing
\[ x_{bf} + x_{br} = w_{bf} + w_{br} \]
\[ x_{cf} + x_{cr} = w_{cf} + w_{cr} \]

This results in 8 equations, and eight unknowns \((x_{br}, x_{bf}, x_{cr}, x_{cf}, p_b, p_c, \lambda_f, \lambda_r)\).

However, some of these equations are redundant. We could remove two of the equations, and drop the variables \(\lambda_f\) and \(\lambda_r\) by replacing the first two optimization equations with the condition that the MRS=price ratio. Also, if we rescale both the prices by the same factor the equilibrium remains unchanged, so without loss of generality we can set \(p_c = 1\). And finally, we can note that since the budget constraints hold for both agents, if one market clearing condition is satisfied the other must be as
well. This is known as Walras’ law. So there are really only 5 equations in 5 unknowns, which we can solve for the relevant variables to define an equilibrium \((x_{br}, x_{bf}, x_{cr}, x_{cf}, p_b)\).

We can now proceed to solve for the equilibrium in this case. Friday’s consumption problem yields condition

\[
\frac{1}{2} \frac{\alpha}{\sqrt{x_{bf}}} = p_b
\]

which implies that

\[
p_b = \alpha \left(\frac{x_{cf}}{x_{bf}}\right)^{\frac{1}{2}}.
\]

and so we can write Friday’s consumption of coconuts as a function of the price and his consumption of bananas

\[
x_{cf} = \left(\frac{p_b}{\alpha}\right)^2 x_{bf}.
\]

And since Robinson has the same preferences we know that

\[
x_{cr} = \left(\frac{p_b}{\alpha}\right)^2 x_{br}.
\]

And using the budget condition

\[
p_b x_{bf} + \left(\frac{p_b}{\alpha}\right)^2 x_{bf} = p_b w_{bf} + w_{cf}
\]

so

\[
x_{bf} = \frac{p_b w_{bf} + w_{cf}}{p_b + \left(\frac{p_b}{\alpha}\right)^2}
\]

and similarly

\[
x_{br} = \frac{p_b w_{br} + w_{cr}}{p_b + \left(\frac{p_b}{\alpha}\right)^2}.
\]

Finally, we must have market clearing for the bananas to have an equilibrium, so we must have

\[
x_{bf} + x_{br} = \frac{p_b w_{bf} + w_{cf}}{p_b + \left(\frac{p_b}{\alpha}\right)^2} + \frac{p_b w_{br} + w_{cr}}{p_b + \left(\frac{p_b}{\alpha}\right)^2} = w_{bf} + w_{br} = w_b
\]

which then implies that

\[
\frac{p_b w_b + w_c}{p_b + \left(\frac{p_b}{\alpha}\right)^2} = w_b
\]

and so

\[
w_c = \left(\frac{p_b}{\alpha}\right)^2 w_b
\]
and so at equilibrium we have that the price is

\[ p_b^* = \alpha \sqrt{\frac{w_c}{w_b}} \]

So we can solve for the equilibrium price in terms of the primitives of the economy. We can then plug this back into the previously found equations both for agents’ consumption and have an expression for consumption in terms of the primitives.

We will conclude this section on competitive equilibrium by reiterating the most famous result in economics, The First Welfare Theorem.

**Theorem 18 (First Welfare Theorem)** Every Competitive Equilibrium is Pareto Efficient.

The idea is that, if we take a competitive equilibrium outcome \((x_f^*, x_r^*)\), and take another allocation \((y_f, y_r)\) where

\[ u_f(y_f) > u_f(x_f^*) \]
\[ u_r(y_r) \geq u_r(x_r^*) \]

this new bundle cannot be to the left of, the right of or on the budget line in the Edgeworth box, and so cannot be feasible.

9 **Producer Theory**

We can use tools similar to those we used in the consumer theory section of the class to study firm behaviour. In that section we assumed that individuals maximize utility subject to some budget constraint. In this section we assume that firms will attempt to maximize their profits given a demand schedule and production technology.

A firm that wishes to produce some output will typically incur some cost of production. The relationship between the amount produced \(q\) and the cost is given by the cost function \(c(q)\). This cost function is itself derived from the production technology. How much a firm produces is a function of the inputs that it uses. Typically the inputs in question are capital \((k)\) and labour \((l)\), so we can write the quantity the firm produces by the relationship \(q = f(k, l)\). The firm then will decide how much \(k\) and \(l\) to use to produce a given level of output. If we assume there is a rental rate of labour of \(w\) (wage rate) and of capital of \(r\) then the cost function can be derived from the minimization problem

\[ c(q) = \min rk + wl \text{ s.t. } f(k, l) = q \]
Example 19 One commonly used production function is the Cobb-Douglas production function where
\[ f(k, l) = k^\alpha l^{1-\alpha} \]

The interpretation is the same as before with \( \alpha \) reflecting the relative importance of capital in production. One nice feature is that doubling inputs also doubles output. This is referred to as constant returns to scale. Hence the resulting cost function is linear, \( c(q) = cq \).

Once the firm knows the cost of producing a given quantity \( q \) it can proceed to determine the optimal quantity it should set. If the firm sells \( q \) units and the price is \( p \) then we can write the firm’s profits as revenue minus cost:
\[ \pi(q) = pq - c(q) \]
and will choose the \( q \) such that marginal revenue equals marginal cost. When the firm is a price taker so that the price does not depend on the quantity the firm provides then the firm chooses \( q \) such that
\[ p = c'(q) \]
so price equals marginal cost.

We can divide the cost into fixed and variable cost, where the fixed cost is the part that does not depend on the amount produced (just as the name suggests). The Average Cost is just the cost divided by the quantity, again obviously. Building a factory or the research necessary to produce a new drug are some examples of fixed costs.

FIGURE 12

9.1 Competitive Industry

If the firm is a price taker then we have seen that the firm will set
\[ p = c'(q) \]

In a competitive market, not only are firms price takers, but there is free entry and exit. Consequently there cannot be positive or negative profits because then there would be entry(exit). So we must also have that
\[ p = AC(q). \]

So the size of the firms in the market is the one that minimizes the average cost, which is where average cost equals marginal cost. Finally, we can invert the cost function to determine how much each firm would produce at each price so that
\[ q = (c')^{-1}(p) \]
and this gives us each firm’s supply curve.
Example 20 Suppose \( c(q) = a + bq^2 \), then \( c'(q) = 2bq \), so the optimal level of quantity each firm produces is \( q = \frac{p}{2b} \).

Example 21 Suppose marginal cost is almost 0. For example, Google, Microsoft, or a prescription drug company which spends billions on research to develop the product but has essentially no marginal cost in producing each additional unit (each search, copy of the software, dose of the drug). Then AC is a decreasing function of quantity. So then it becomes efficient to have a few large firms or even a monopoly. However, then it doesn’t make sense that the firm would be a price taker.

9.2 Pricing Power

If a firm produces a non-negligible amount of the overall market then the price at which the good sells will depend on the quantity sold. For any given price there will be some quantity demanded by consumers, and this is known as the demand curve. We can invert this relationship to get the Inverse Demand function \( p(q) \) which reveals the price that will prevail in the market if the output is \( q \). If the firm is a monopolist then their goal is to maximize

\[
\pi(q) = p(q)q - c(q)
\]

where \( p'(q) < 0 \) since less will be sold at a higher price. A simple example used frequently is \( p(q) = a - bq \), and we will often assume that \( c(q) = cq \) just for the sake of simplicity (Cobb-Douglas production provides this for example). We will also assume that \( a > c \) since otherwise the cost of producing is higher than any consumer’s valuation so it will never be profitable for the firm to produce and the market will cease to exist. Then the firm want to maximize the objective

\[
q(a - bq - c).
\]

FIGURE 13

The efficient quantity is produced when \( p(q) = a - bq = c \) because then a consumer buys an object if and only if they value it more than the cost of producing, resulting in the highest possible total surplus. So the efficient quantity is

\[
q^* = \frac{a - c}{b}.
\]

The monopolist’s maximization problem, however, has FOC

\[
a - 2bq - c
\]
where $a - 2bq$ is the marginal revenue and $c$ is the marginal cost. So the quantity set by the monopolist is

$$q^M = \frac{a - c}{2b} < q^*.$$ 

The price with a monopoly can easily be found since

$$p^M = a - bq^M$$
$$= a - \left(\frac{a - c}{2}\right)$$
$$= \frac{a + c}{2} > c.$$ 

Why isn’t the marginal revenue equal to the price? Because selling more requires setting a lower price which results in lower revenue on the other sales. In general, when the problem is to maximize $p(q)q - c(q)$ the associated FOC is

$$p(q) + p'(q)q - c'(q) = 0$$

where $p(q)$ reflects the revenue from the marginal buyer, and $p'(q)q (<0)$ reflects the lost revenues by being forced to sell to other buyers at a lower price. So $p(q) + p'(q)q$ is the marginal revenue, and it must equal the marginal cost.

### 9.3 Price Discrimination

First degree price discrimination (perfect price discrimination) means discrimination by the identity of the person or the quantity ordered (non-linear pricing). It will result in an efficient allocation. Suppose there is a single buyer and a monopoly seller where the inverse demand is given by $p = a - bq$. If the monopolist were to set a single price they would set the monopoly price. As we saw in the previous section, however, this does not maximize the joint surplus, so the monopolist can do better. Suppose instead that the monopolist charges a fixed fee $F$ to be allowed to buy, and then sells at a price $p$, and suppose they set price $p = c$. The fixed fee will not affect the quantity that a participating consumer will choose, so if the consumer participates then they will choose quantity equal to $q^*$. The firm can then set the entry fee to extract all the consumer surplus and the consumer will still be willing to participate. This maximizes the joint surplus, and gives the entire surplus to the firm, so the firm is doing as well as it could under any other mechanism. Specifically the firm sets

$$F = \frac{(a - c)q^*}{2} = \frac{(a - c)^2}{2b}.$$
(in integral notation this is $F = \int_0^{q^*} (p(q) - c) dq$). This pricing mechanism is called a Two-Part Tariff, and was famously used at Disneyland (entry fee followed by a fee per ride), greatly increasing revenues.

FIGURE 14

Second degree price discrimination occurs when the firm cannot observe to consumer’s willingness to pay directly. Consequently they elicit these preferences by offering different quantities or qualities at different prices. The consumer’s type is revealed through which option they choose. This is known as screening.

Suppose there are two types of consumers. One with high valuation, and one with low. $v_H > v_L > 0$. Let $p$ denote the fraction of consumers who have the high valuation. Assume that the firm can produce a product of quality $q$ at cost $c(q) = \frac{1}{2} q^2$. If a consumer of type $i$ buys an object of quality $q$ for price $t$ then they get payoff of

$$u_i(q, t) = v_i q - t$$

and the firm earns profits of

$$t - \frac{1}{2} q^2$$

from selling an object of quality $q$ for price $t$. Assume that the consumer receives payoff of 0 if they do not purchase any object.

If the firm knew the type each consumer they could offer a different quality to each consumer. The quality that maximizes the social surplus for each type would maximize

$$u_i(q_i, t_i) + \pi_i(q_i, t_i) = v_i q_i - t_i + t_i - \frac{1}{2} q_i^2 = v_i q_i - \frac{1}{2} q_i^2$$

which is maximized by setting

$$q_i = v_i$$

for each type. In many situations, the firm will not be able to observe the valuation of the consumer. In such a situation the firm offers a schedule of price-quality pairs and let the consumers self-select into contracts. This is typically referred to Adverse Selection from the Insurance industry where individuals who are the worst risks are the ones who select into the most extensive coverage.

Since there are two types of consumers the firm will offer two different quality levels, one for the high valuation consumers and one for the low
valuation consumers. Hence there will be a choice of two contracts \((t_H, q_H)\) and \((t_L, q_L)\) the first bought by the high valuation consumers, the second by the low valuation consumers. The consumers’ have the option of walking away, so the firm cannot demand payment higher than the value of the object. That is we must have

\[
\begin{align*}
v_H q_H - t_H & \geq 0 \\
v_L q_L - t_L & \geq 0
\end{align*}
\]

These are known as the Individual Rational (IR) constraints that guarantee the consumers are willing to participate.

Since we know that the efficient qualities are \(q_i = v_i\) for both types, and that the firm does not want to leave consumers with any unnecessary surplus, we might expect that the firm would \(q_i = v_i\) and \(t_i = v_i^2\). However, this leaves the consumers with no surplus. If the high valuation consumer bought the lower quality product they would receive payoff of \(v_H q_L - t_L = q_L (v_H - v_L) > 0\). So the firm must structure the contracts so that consumers do not want to mis-report their type. That is, they must set the contract such that

\[
\begin{align*}
v_H q_H - t_H & \geq v_H q_L - t_L \\
v_L q_L - t_L & \geq v_L q_H - t_H
\end{align*}
\]

These are known as the Incentive Compatibility (IC) constraints that guarantee that consumers select into the right pool.

This leaves 4 constraints that need to be satisfied, but it is easy to verify that the other two conditions are satisfied whenever

\[
\begin{align*}
v_L q_L - t_L & \geq 0 \\
v_H q_H - t_H & \geq v_H q_L - t_L
\end{align*}
\]

And since the firm will not give the consumers any more then they have to, it is optimal to set

\[
\begin{align*}
t_L & = v_L q_L \\
t_H & = v_H q_H - q_L (v_H - v_L)
\end{align*}
\]

We wont solve for the optimal quality here (although the problem is doable with the tools we have developed so far), but this analysis indicates some important results about second degree price discrimination:

1. The low type receives no surplus.
2. The high type receives a positive surplus of \(q_L (v_H - v_L)\). This is known as an informational rent, that the consumer can extract because the seller does not know their type.
3. The firm should set the efficient quality $q_H = v_H$ for the high valuation type.

4. The firm will degrade the quality for the low type ($q_L < v_L$) so as to lower the rents the high type consumers can extract.

9.4 Bundling

Another way in which a firm can increase its profits is through bundling. Rather than selling an object individually it is packaged with another object. To see why this may be profitable, consider the following example.

Example 22 Suppose there are two types of consumers A and B, each accounting for half the market. Suppose the firm has two products to sell, a spreadsheet and a word processor. Suppose the type-A consumers value the spreadsheet at 150 and the word processor at 100, and the type-B consumers value the spreadsheet at 100 and the word processor at 150. Suppose the marginal cost is 0.

Suppose the firm was pricing the word processor individually. By setting price 100 (or below) they sell to everyone, pricing at 150 they sell to $\frac{1}{2}$ the market, and at any higher price they sell to none. So the optimal price is 100, earning revenues of 100 per consumer. Similarly, the price of the spreadsheet is 100, and the total profits of the firm is 200 per consumer.

Suppose instead that the firm set a price for BOTH the spreadsheet and the word processor. Both types have valuation of 250 for both items together, so the firm can set price of 250 for the bundle and earn profits 250 per consumer, higher than when the goods are price separately.

A couple of things to note about the example. If there had been another type that didn’t value one of the objects at all, the example could still go through where the firm offers a price for each item individually and a price for the bundle that is lower then the price for each good individually (i.e. offer a discount to buy both). Secondly, this example is very stark in that there is a negative correlation between the valuations of the two products, which seems unlikely in most settings. In general, this will not be necessary for bundling to be profitable, and an example with profitable bundling where valuations are independent is given in problem set 9.

9.5 Oligopoly

Oligopoly refers to environments where there are few large firms. These firms are large enough that their quantity influences the price and so
impacts their rivals. Consequently each firm must condition its behaviour on the behaviour of the other firms. This strategic interaction is modelled with game theory.

Probably the most important model of Oligopoly is the Cournot model of quantity competition. We will first consider the duopoly case, where there are only two firms. Suppose the inverse demand function is given by \( p(q) = a - bq \), and the cost of producing is constant and the same for both firms \( c_i(q) = cq \). The quantity produced in the market is the sum of what both firms produce \( q = q_1 + q_2 \). The profits for each firm is then a function of the market price and their own quantity,

\[
\pi_i(q_1, q_2) = q_i(p(q) - c)
\]

The strategic variable that the firm is choosing is the quantity to produce \( q_i \).

Suppose that the firms’ objective was to maximize their joint profit

\[
\pi_1(q_1, q_2) + \pi_2(q_1, q_2) = (q_1 + q_2)(p(q_1 + q_2) - c)
\]

then we know from before that this is maximized when \( q_1 + q_2 = q^M \). We could refer to this as the collusive outcome. One example would be \( q_1 = q_2 = \frac{q^M}{2} \).

If the firms could write binding contracts then they could agree on this outcome. However, that typically won’t be possible (such an agreement would be price fixing), so we would not expect this outcome to occur unless it is stable/self-enforcing. If either firm could increase its profits by setting another quantity, then they would have an incentive to deviate from this outcome. We will see below that both firms would in fact have an incentive to deviate and increase their output.

Suppose now that firm \( i \) is trying to choose \( q_i \) to maximize its own profits, taking the other firm’s output as given. Then firm \( i \)’s problem is to maximize

\[
\pi_i(q_1, q_2) = q_i(a - b(q_1 + q_2) - c)
\]

which has associated FOC

\[
\frac{\partial \pi_i(q_1, q_2)}{\partial q_i} = a - c - bq_j - 2bq_i = 0
\]

So then the optimal level of \( q_i \) for any level of \( q_j \) is

\[
q^*_i(q_j) = \frac{a - bq_j - c}{2b}
\]

This is \( i \)’s best response to whatever \( j \) plays. In the special case when \( q_j = 0 \) the firm is a monopolist, and the observed quantity corresponds
to the monopoly case. In general, when the rival has produced $q_j$ we can treat the firm as a monopolist facing a "residual demand curve" with intercept of $a - bq_j$. We can write firm i’s best response function as

$$q_i^*(q_j) = \frac{a - c}{2b} - \frac{1}{2}q_j$$

so

$$\frac{dq_i}{dq_j} = -\frac{1}{2}.$$

This has two important implications. First, the quantity player i chooses is decreasing in its rival’s quantity. This means that quantities are strategic substitutes. Second, if player j increases their quantity player i decreases their quantity by less than player j increased their quantity. So we would expect that the output in a duopoly would be higher than in a monopoly.

We are at a "stable" outcome if both firms are producing a best response to their rivals’ production. We refer to such an outcome as an equilibrium. That is, when

$$q_1 = \frac{a - bq_2 - c}{2b}$$

$$q_2 = \frac{a - bq_1 - c}{2b}$$

since the best responses are symmetric we will have $q_1 = q_2$ and so we can calculate the equilibrium quantities from the equation

$$q_1 = \frac{a - bq_1 - c}{2b}$$

and so

$$q_1 = q_2 = \frac{a - c}{3b}$$

and so

$$q = q_1 + q_2 = \frac{2(a - c)}{3b} > \frac{a - c}{2b} = q^M$$

so there is a higher output (and hence lower price) in a duopoly than a monopoly as expected.

FIGURE 15.

More generally, both firms are playing a best response to their rival’s action because

$$\pi_i(q_1^*, q_2^*) \geq \pi_i(q_1, q_2^*)$$ for all $i, q_i$

That is, the profits from the quantity are (weakly) higher then the profits from any other output. This motivates the following definition for an equilibrium in a strategic setting.
Definition 23 If \((q^*_1, q^*_2)\) satisfies

\[
\pi_i(q^*_1, q^*_2) \geq \pi_i(q_1, q^*_2) \quad \text{for all} \quad i, q_i
\]

then \((q^*_1, q^*_2)\) is a Nash Equilibrium.

Nash Equilibrium is ultimately a stability property. There is no profitable deviation for any of the players. In order to be at equilibrium we must have that

\[
q_2 = q^*_2(q_1) \\
q_1 = q^*_1(q_2)
\]

and so we must have that

\[
q_1 = q^*_1(q^*_2(q_1))
\]

so equilibrium corresponds to a fixed-point of the mapping \(q^*_1(q^*_2(\cdot))\).

9.6 First Mover Advantage

The above analysis assumed that the situation was static, both firms set quantities simultaneously, and care only about their current profits. This implicitly assumes there will be no future interaction between the players where past behaviour is relevant. We can use the same tools to consider dynamic situations, where one firm sets its quantity first and then the second firm responds. The most natural interpretation is that the leader is the incumbent in the market, and the second firm is a new entrant. What is important here is firm 1 commits to a quantity before firm 2 selects its quantity. Since firm 1 must take into account the effect of their quantity on firm 2, we use "backward induction" to solve this game. We first determine what firm 2 will produce in response to any quantity from quantity 1 in the first period. The first firm takes firm 2's response to its quantity into account when it sets its quantity in the first period. If we assume, just for simplicity, that marginal cost is 0 then in the second period firm 2 chooses \(q_2\) to maximize

\[
\pi_2(q_1, q_2) = (a - b(q_1 + q_2))q_2
\]

and so we have the FOC

\[
\frac{d\pi_2}{dq_2} = a - bq_1 - 2bq_2 = 0
\]

and so 2's best response is to set

\[
q_2(q_1) = \frac{a - bq_1}{2b}
\]
and again we see that

\[ q_2'(q_1) = -\frac{1}{2} < 0 \]

so again we see that \( q_1 \) and \( q_2 \) are strategic substitutes.

Now in the first period, the leader knows how their rival will respond to whatever quantity they produce and will take this into account when setting quantity. So firm 1 chooses quantity to maximize

\[
\hat{\pi}_1(q_1) = \pi_1(q_1, q_2(q_1))
\]

\[ = (a - b(q_1 + q_2(q_1)))q_1 \]

\[ = (a - b(q_1 + \frac{a - bq_1}{2b}))q_1 \]

\[ = (\frac{a}{2} - \frac{b}{2}q_1)q_1 \]

the maximization problem then results in the FOC

\[ \frac{d\hat{\pi}_1}{dq_1} = \frac{a}{2} - bq_1 = 0 \]

and so then

\[ q_1^* = \frac{a}{2b} = q_M > q_1^C \]

and so

\[ q_2^* = \frac{a - bq_1^*}{2b} = \frac{a - \frac{a}{2}}{2b} = \frac{a}{4b} < q_2^C \]

So we notice that firm 1 exploits its first mover advantage to produce more than its rival. Notice also that \( q_1 + q_2 = \frac{3a}{4b} \) which is higher then the total output of the Cournot game. This is because firm 1, anticipating its affect on its rival, produces more than the Cournot output, and firm 2 only cuts back production \( \frac{1}{2} \) for each addition unit firm 1 produces.

The key here was that firm 1 was able to commit to the high output. If it could go back and adjust its output at the end it would undermine this commitment and remove the first mover advantage. So the restriction that the firm cannot go back and adjust its output is actually beneficial. In the single agent decision problem constraints are never beneficial, but in a strategic environment they may be. Firm 2 which is at a disadvantage could try to threaten to produce more to try to prevent firm 1 from producing so much, but such a claim would not be credible (since if firm 2 does not play a best response it decreases its own profits). Of course, we have assumed that the follower is motivated solely by the desire to maximize its profits in this period (i.e. no future periods where their behaviour this period can influence the future payoff).
9.7 Competition on Price

So far we have assumed that firms compete on the basis of quantity. However, the relevant strategic variable could also be the price set and not the quantity. In the market with inverse demand \( p = a - bq \) the demand in the market at any price is simply \( q = \frac{a}{1+b} \). Suppose there are two firms that produce homogenous goods, that both firms have the same marginal cost \( c \), and that both firms have capacity sufficiently large to serve the entire market. Define \( p_{\min} = \min\{p_1, p_2\} \). Assume that if one firm sets a lower price it sells to the entire market, and if both firms set the same price they each sell to half the market. Then the profit of firm \( i \) is

\[
\pi_i(p_1, p_2) = \begin{cases} 
    \left( \frac{a-p_i}{b} \right) (p_i - c), & p_i < p_j \\
    0, & p_i > p_j \\
    \frac{1}{2} \left( \frac{a-p_i}{b} \right) (p_i - c), & p_i = p_j
\end{cases}
\]

FIGURE 16.

Notice that the profit function is not continuous and so is not differentiable. So we cannot use FOCs to determine the optimal solution. However, it is straightforward to verify that the resulting equilibrium is

\[ p_1 = p_2 = c. \]

It is obvious that this is an equilibrium, since neither firm has an incentive to increase price (sell 0 so earn 0 profits) or decrease price (price below cost means negative profits). To see that this is the only equilibrium, we just have to rule out the other possibilities. Clearly, it can never be optimal for either firm to set \( p_i < c \). We can’t have an equilibrium with \( p_1 = p_2 > c \) since then by undercutting their rival slightly they almost double their profits. So we would have to look for an equilibrium where \( p_1 \neq p_2 \). Without loss of generality assume that \( p_1 < p_2 \). This is not possible since if \( p_1 = c \) firm 1 could increase its price slightly and earn positive profits, and if \( p_1 > c \) then firm 2 could set \( p_2 \in (c, p_1) \) and earn positive profits. So we can exhaustively rule out all other possibilities.

9.8 Beauty Contest

An example of a game is a "beauty contest". Everyone in the class picks a number on the interval \([0, 100]\). The goal is to guess as close as possible to \( \frac{2}{3} \) the class average. An equilibrium of this game is for everyone to guess 0. This is in fact the only equilibrium. Since noone can guess more than 100, \( \frac{2}{3} \text{mean} \) cannot be higher than \( 66\frac{2}{3} \), so all guesses above this are "dominated". But since noone will guess more than \( 66\frac{2}{3} \) the mean cannot be higher than \( \frac{2}{3} \times (66\frac{2}{3}) = 44\frac{4}{9} \), so noone should guess...
higher than $44\frac{4}{5}$. Repeating this $n$ times noone should guess higher than $(\frac{2}{3})^n100$ and taking $n->\infty$ all players should guess 0. Of course, this isn’t necessarily what will happen in practice if people solve the game incorrectly or expect others too. Running this experiment in class the average guess was approximately 13.5.

9.9 Hotelling’s Linear City

Hotelling(1930) introduced a model of linear city as a simple model of differentiated products. If the same object is available in two different locations, these are in essence two different products, since with transportation costs different customers have different preferences for buying from different firms (just as different consumers would have different preferences for differentiated products). This allows firms some degree of pricing power.

Suppose the inhabitants of a city are uniformly distributed along a line from 0 to 1. Firm 1 is located at 0, and firm 2 is located at 1. Assume all consumers have valuation $v$ for an object (and only demand 1) and that the marginal cost of producing said object is 0. Suppose that a customer must pay cost $c$ proportional to the distance travelled when they buy an object. So if consumer $i$ buys from firm $j$ at price $p_j$ then their utility is

$$v - cd_{ij} - p_j$$

where $d_{ij}$ is the distance between person $i$ and firm $j$.

Two extreme cases are when $c = 0$ and when $c \geq 2v$. In the first case there are no transaction costs so the firms are not differentiated, so we are left with Bertrand competition resulting in $p = 0$. In the second case, no consumer above (below) $\frac{1}{2}$ would ever buy from firm 1 (resp. firm 2) no matter the price set. So the firms aren’t competing for any of the same consumers, and consequently are acting as monopolists in a local market.

Suppose we are at an intermediate value for $c$. Suppose consumer $i$, where $i \in [0,1]$ is indifferent between buying from firm 1 and firm 2. Then all consumers to the left of $i$ buy from firm 1 and all consumers to the right buy from firm 2, so firm 1 is selling to fraction $i$ of the market. If the prices set by the firms are $(p_1, p_2)$ then we can derive the marginal consumer as

$$v - ci^* - p_1 = v - c(1 - i^*) - p_2$$

since the distance from consumer $i$ to firm 1 is $i$, and the distance to firm 2 is 1-i. So

$$2i^*c = c - p_1 + p_2$$
and 
\[ i^* = \frac{1}{2} + \frac{1}{2c}(p_2 - p_1) \]

So taking firm 2’s price as given firm 1’s profits by setting price \( p_1 \) is

\[ \pi_1(p_1, p_2) = p_1\left(\frac{1}{2} + \frac{1}{2c}(p_2 - p_1)\right) \]

and similarly

\[ \pi_2(p_1, p_2) = p_2\left(\frac{1}{2} + \frac{1}{2c}(p_1 - p_2)\right) \]

We now look for the Nash equilibrium \((p_1^*, p_2^*)\) where both firms are setting the optimal price given their rival’s price. So we can derive Firm 1’s best response as

\[ \frac{\partial \pi_1(p)}{\partial p_1} = \frac{1}{2} + \frac{1}{2c}(p_2 - 2p_1) = 0 \]

so

\[ p_1^*(p_2) = \frac{1}{2}(c + p_2) \]

and notice that

\[ p_1^*(p_2) = \frac{1}{2} > 0 \]

so prices are strategic substitutes. We can similarly derive an expression for 2’s best response

\[ p_2^*(p_1) = \frac{1}{2}(c + p_1) \]

and then solve for the Nash Equilibrium. By symmetry we know that \( p_1^* = p_2^* = p^* \) so

\[ p^* = \frac{1}{2}(c + p^*) \]

and so

\[ p^* = c. \]

So the price is higher the greater the degree of differentiation. Notice also that \( i^* = \frac{1}{2} \), so the firms will split the market regardless of the cost. Finally, notice that we have assumed that the surplus from buying for the marginal consumer is non-negative, so we must have

\[ v - \frac{1}{2}c - c = v - \frac{3}{2}c \geq 0. \]

The linear city is an example of horizontal differentiation, where different consumers have different rankings of the value of the different
products. We could also consider vertical differentiation, where all consumers agree on the relative ranking of the products, but have different willingness to pay. Suppose the product in question is of quality \( q \) and consumer \( i \) has valuation \( iq \) for a product of quality \( q \). Now consider the following game: First the firms decide on the \( q_i \) to produce, then the firms set prices. The key here is that firms will not locate at the same qualities, since then Bertrand competition will guarantee that neither firm will earn any profits. Instead one firm will locate at a high quality. This firm will then set a high price and sell to the section of the market with the highest valuations. The other firm will choose a low quality, and sell at a low price to consumers unwilling to buy the high quality good.

9.10 Collusion

So far we have assumed that games we have analyzed are "one-shot" games where the players will never interact again, and so there is no possibility for future rewards or punishment. While there are many such situations, there are many other situations where the players will be interacting with each in the future. In such situations each player can condition their play in future periods on how the game as played in the current period. This will typically allow for outcomes that are not possible in the static game to be achieved. While a formal analysis of repeated games is beyond the scope of this course we will provide a simple example of how collusion can be sustained if the firms will interact again in the future. We will assume that the game is repeated infinitely often, but the value of payoffs in future periods is discounted relative to the current period.

Suppose that if firms collude (and jointly behave like a monopolist) then each firm will earn profits \( \pi^m \). We have seen that in a one-shot game this is not sustainable as each firm could gain by undercutting, and we’ll denote the profits from undercutting or deviating as \( \pi^d \). Finally, assume that without colluding the firms receive profits \( \pi^c \) in the one-shot Nash Equilibrium. Since the collusive outcome increases profits

\[
\pi^d > \pi^m > \pi^c
\]

Now suppose that both firms play the following strategies (known as grim trigger):

1. Set the monopoly output in the first period.
2. Continue playing the monopoly output until someone has deviated. As soon as one player has deviated (even once), play the competitive outcome forever.
This strategy uses a "carrot" (reward cooperation with future cooperation) and a "stick" (deviation is punished with competition in the future). If a firm thought its rival was playing this strategy, then by following this strategy and producing the monopoly output the monopoly profits are earned in every period forever. If we assume that there is a discount factor of $\delta$ (so that payoffs tomorrow are worth $\delta < 1$ as much as payoffs today, payoffs in two periods are worth $\delta^2$ as much,...) then the profits from this strategy are $\frac{\pi^m}{1-\delta}$. Now suppose the firm deviates. Then the competitive outcome is achieved in all future periods. Then it will get $\pi^d$ today and $\pi^c$ in every future period (not in the current period). So the profits from deviating are $\pi^d + \frac{\delta}{1-\delta} \pi^c$. So the firm will be willing to collude today if

$$\frac{\pi^m}{1-\delta} \geq \pi^d + \frac{\delta}{1-\delta} \pi^c$$

and since the decision problem in each period is identical, if the firm is willing to collude in this period it will be willing to collude in every future period as well. So when the above inequality holds collusion will be sustainable indefinitely. That is, collusion will be possible when the gains from future cooperation is higher than the benefit today from undercutting. This is possible if $\delta$ is large enough (the firms place a high enough value on future profits). If the firm is myopic ($\delta = 0$) then collusion is impossible.

### 10 Game Theory

In the previous section we introduced game theory in the context of firm competition. We will conclude this course by considering more general games and other important applications, such as auctions. The specification of (static) game consists of three elements: the players, the strategies available, and the payoffs for each player as a function of the strategies of the players. We use game theory to analyze situations where there is strategic interaction so the payoff function will typically depend on the strategies of other players as well. Typically we will index the players by $i = 1, ..., I$ and we assume that each player chooses strategy $s_i$ from the available strategies $S_i$. We can write $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_I)$ to represent the strategies of the other $I-1$ players, and write the payoff function for player $i$ as $u_i(s_i, s_{-i})$.

**Example 24** In the duopoly Cournot example $I = 2$, $S_i = R_+$ and the payoffs are just the profits so $u_i(s_1, s_2) = \pi_1(s_1, s_2) = s_i(a-b(s_1+s_2)-c).$ Here the interpretation of the strategy is the quantity chosen, which can be any non-negative number.
We can represent games (at least those with a finite choice set) in Normal form. A normal form game consists of the matrix of payoffs for each player from each possible strategy. If there are two players, 1 and 2, then the normal form game consists of a matrix where the \((i, j)\)th entry consists of the tuple (Player 1’s payoff, Player 2’s payoff) when player 1 plays their \(i\)th strategy and player 2 plays their \(j\)th strategy. We will now consider the most famous examples of games.

**Example 25 (Prisoner’s Dilemma)** Suppose two suspects, Bob and Rob are arrested for a crime and questioned separately. The police can prove the committed a minor crime, and suspect they have committed a more serious crime but can’t prove it. The police offer each suspect that they will let them off for minor crime if they confess and testify against their partner for the more serious crime. Of course, if the other criminal confesses the police won't need his testimony but will give him a slightly reduced sentence for cooperating. Each player then has two possible strategies: Stay Quiet \((Q)\) or Confess \((C)\) and they decide simultaneously. We can represent the game with the following payoff matrix:

\[
\begin{array}{c|cc}
 & Q & C \\
\hline
B\!ob & (3, 3) & (-1, 4) \\
C & (4, -1) & (1, 1) \\
\end{array}
\]

Each entry represents \((Bob, Rob)\)'s payoff from each of the two strategies. For example, if Rob stays quiet while Bob confesses Bob's payoff is 4 and Rob's is -1. Notice that both players have what's known as a dominant strategy, they should confess regardless of what the other player has done. If we consider Bob, if Rob is Quiet then confessing gives payoff \(4 > 3\) the payoff from staying quiet. If Rob confesses, then Bob should confess since \(1 > -1\). The analysis is the same for Rob. So the only stable outcome is for both players to confess. So the only Nash Equilibrium is \((Confess, Confess)\). Notice that, from the perspective of the Prisoners this is a bad outcome. In fact it is pareto dominated by both players staying quiet, which is not a Nash equilibrium.

The above example has a dominant strategy equilibrium, where both players have a unique dominant strategy.

**Definition 26** A strategy is \(s_i\) is dominant if

\[u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}) \text{ for all } s'_i \in S_i, s_{-i} \in S_{-i}\]

If each player has a dominant strategy, then the only rational thing for them to do is to play that strategy, so if a dominant strategy equilibrium exists it is a relatively uncontroversial prediction of what will
happen in the game. However, it is rare that a dominant strategy will exist in most strategic situations. Consequently, the most commonly used solution concept is Nash equilibrium. We’ve previously defined Nash Equilibrium with firms. The definition extends to general games.

**Definition 27** A strategy \( s^* = (s^*_1, ..., s^*_I) \) is a Nash Equilibrium if for all \( i \)

\[
s^*_i \in \arg \max_{s_i} \{ u_i(s_i, s^*_{-i}) \}.
\]

We refer to strategy \( s^*_i \) as a best response to \( s^*_{-i} \).

To reiterate, a strategy profile is a Nash equilibrium if each player is playing a best response to the other players’ strategies. So a Nash equilibrium is a stable outcome where no player could profitably deviate. Clearly when dominant strategies exist it is a Nash equilibrium for all players to play a dominant strategy. However, as we see from the Prisoner’s Dilemma example the outcome is not necessarily efficient. The next example shows that the Nash Equilibrium may not be unique.

**Example 28** (Coordination Game) We could represent a coordination game where Bob and Ann choose between the Ballet and the Theatre, where both prefer the Ballet but want go to the same place.

<table>
<thead>
<tr>
<th></th>
<th>Bob</th>
<th>Ann</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td></td>
<td>(2, 2)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td></td>
<td>(0, 0)</td>
<td>(1, 1)</td>
</tr>
</tbody>
</table>

Here (B,B) and (T,T) are both equilibria. Notice that the equilibria in this game are pareto ranked with both players preferring to coordinate on the ballet. Both players going to the theatre is also an equilibrium, since if both players think the other will play T they will play T as well.

A famous example of a coordination game is from traffic control. It doesn’t really matter if everyone drives on the left or right, as long as everyone drives on the same side.

So far we have considered only pure strategies: strategies where the players do not randomize over which action they take. The following simple example demonstrates that a pure strategy Nash Equilibrium may not always exist.

**Example 29** (Matching Pennies) Consider the following payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>Bob</th>
<th>Ann</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>H</td>
<td>H</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td></td>
<td>(1, -1)</td>
<td>(1, -1)</td>
</tr>
<tr>
<td></td>
<td>(-1, 1)</td>
<td>(-1, 1)</td>
</tr>
</tbody>
</table>
Here Ann wins if both players play the same strategy, and Bob wins if they play different ones. Clearly there cannot be pure strategy equilibrium, since Bob would have an incentive to deviate whenever they play the same strategy and Ann would have an incentive to deviate if they play the differently. Suppose instead that Ann plays H with probability $p$. Then Bob’s payoff from playing H is

$$-p + (1 - p) = 1 - 2p$$

and his payoff from playing T is

$$p - (1 - p) = 2p - 1$$

So when $p = \frac{1}{2}$ Bob is indifferent between playing H or playing T. So a best response by Bob is to randomize between H and T. Similarly, Ann is willing to randomize if Bob is playing H with probability $p = \frac{1}{2}$. So we have that the equilibrium of the above game where both players play H with probability $\frac{1}{2}$ and T with probability $\frac{1}{2}$.

While the idea of a matching pennies game may seem contrived, it is merely the simplest example of a general class of zero-sum games, where the total payoff of the players is constant regardless of the outcome. Consequently gains for one player can only come from losses of the other. For this reason, zero-sum games will rarely have a pure strategy Nash equilibrium. Examples would be chess, or more relevantly, competition between two candidates or political parties. Cold War power politics between the US and USSR was famously (although probably not accurately) modelled as a zero-sum game. Most economic situations are not zero-sum since resources can be used inefficiently.

A slight variation is the game of Rock-Paper-Scissors.

<table>
<thead>
<tr>
<th></th>
<th>Bob</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Ann</td>
<td>R</td>
<td>(0, 0)</td>
<td>(−1, 1) (1, −1)</td>
</tr>
<tr>
<td></td>
<td>P</td>
<td>(1, −1) (0, 0) (−1, 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>S</td>
<td>(−1, 1) (1, −1) (0, 0)</td>
<td></td>
</tr>
</tbody>
</table>

The reader should be able to verify that unique Nash equilibrium is for both players to play each strategy with probability $\frac{1}{3}$. Notice that the expected payoffs for both players is 0.

### 10.1 Auctions

We now proceed to an area that Game Theory has been extremely successful in understanding, that of auctions. In an auction there are I bidders, and each bidder $i$ from 1 to I has a valuation $v_i$ for a single object. If the bidder wins the object at price $p_i$ then they receive utility $v_i - p_i$. Let $b_i$ denote the bid of player $i$. 
Auctions can be either sealed bid or open bid. Examples of sealed bid auctions are the first price auction (where the winner is the bidder with the highest bid and they pay their bid), and the second price auction (where the bidder with the highest bid wins the object and pays the second highest bid as a price). Open bid auctions include English auctions (the auctioneer sets a low price and keeps increasing the price until all but one player has dropped out) and the Dutch auction (a high price is set and the price is gradually lowered until someone accepts the offered price). In many situations (specifically when the other players’ valuations does not affect your valuation) the optimal behaviour in a second price auction is equivalent to an English auction, and the optimal behaviour in a first price auction is equivalent to a Dutch auction. This provides a motivation for considering the second price auction which is strategically very simple, since the English auction is commonly used. It’s the mechanism used in the auction houses, and is a good first approximation how auctions are run on eBay.

How should people bid in a second price auction? Typically a given bidder will not know the bids/valuations of the other bidders. A nice feature of the second price auction is that the optimal strategy is very simple and does not depend on this information: each bidder should bid their true valuation.

**Proposition 30** In a second price auction it is a Nash Equilibrium for all players to bid their valuations. That is $b^*_i = v_i$ for all $i$ is a Nash Equilibrium.

**Proof.** Without loss of generality, we can assume that player 1 has the highest valuation. That is, we can assume $v_1 = \max_i \{v_i\}$. Similarly, we can assume without loss of generality that the second highest valuation is $v_2 = \max_{i>1} \{v_i\}$.

Define

$$
\mu_i(v_i, b_i, b_{-i}) = \begin{cases} v_i - p_i, & \text{if } b_1 = \max_j \{b_j\} \\ 0, & \text{otherwise} \end{cases}
$$

to be the surplus generated from the auction for each player $i$. Then under the given strategies ($b = v$)

$$
\mu_i(v_i, v_i, v_{-i}) = \begin{cases} v_1 - v_2, & i = 1 \\ 0, & \text{otherwise} \end{cases}
$$

So we want to show that no bidder has an incentive to deviate. First we consider player 1. The payoff from bidding $b_1$ is

$$
\mu_1(v_1, b_1, v_{-1}) = \begin{cases} v_1 - v_2, & \text{if } b_1 > v_2 \\ 0, & \text{otherwise} \end{cases} \leq v_1 - v_2 = \mu_1(v_1, v_1, v_{-1})
$$
do player 1 cannot benefit from deviating.

Now consider any other player $i > 1$. They win the object only if they bid more than $v_1$ and would pay $v_1$. So the payoff from bidding $b_i$ is

$$
\mu_i(v_i, b_i, v_{-i}) = \begin{cases} 
  v_i - v_1, & \text{if } b_i > v_1 \\
  0, & \text{otherwise}
\end{cases}
$$

since $v_i - v_1 \leq 0$. So player i has no incentive to deviate either.

We have thus verified that all players are choosing a best response, and so the strategies are a Nash Equilibrium. ■

11 Asymmetric Information

11.1 Adverse Selection

We have touched on asymmetric information already in this course. As it sounds like, asymmetric information simply refers to situations where some of the players have relevant information that other players do not. In the section on price discrimination we assumed (not unreasonably) that buyers know their own valuation but the seller does not (i.e. an individual buyer could have a high or low valuation for the object). We now consider a famous example where that asymmetry is reversed: the market for lemons.

Example 31 Suppose there is a potential buyer and a potential seller for a car (or other object). Suppose that the quality of the car is denoted by $\theta \in [0, 1]$. Buyers and sellers have different valuations/willingness to pay $v_b$ and $v_s$, so that the value of the car is $v_b \theta$ to the buyer and $v_s \theta$ to the seller. Assume that $v_b > v_s$ so that the buyer always values the car more highly then the seller. So we know that trade is always efficient. Suppose that both the buyer and seller know $\theta$, then we have seen in the bilateral trading section that trade can occur at any price $p \in [v_s \theta, v_b \theta]$ and at that price the efficient allocation (buyer gets the car) is realized (buyer gets $v_b \theta - p$, seller gets $p - v_s \theta$ and the total surplus is $v_b \theta - v_s \theta$).

The assumption that the buyer knows the quality of the car may be reasonable in some situations (new car), but in many situations the seller will be much better informed about the car’s quality. The buyer of a used car can observe the age, mileage, etc of a car and so have a rough idea as to quality, but the seller has presumably been driving the car and will know more about it (this asymmetry can presumably be ameliorated though not eliminated by mechanical inspections though one could come up with many other examples where such inspections may not be feasible). In such a situation we could consider the quality $\theta$ as a random variable, where the buyer knows only the distribution but
the seller knows the realization. We could consider a situation where
the buyer know the car is of a high quality with some probability, and
low quality otherwise, whereas the seller knows whether the car is high
quality. Obviously the car could have a more complicated range of
potential qualities. If the seller values a high quality car more, then
their decision to participate in the market potentially reveals negative
information about the quality, hence the term adverse selection. How
does this change the outcome?

Example 32 Suppose instead that the buyer only knows that \( \theta \sim U[0, 1] \).
That is that the quality is uniformly distributed between 0 and 1. Then
the buyer’s willingness to pay is simply

\[
E[v_b \theta] = v_b E[\theta] = \frac{v_b}{2}.
\]

Since \( v_b \) is a constant with respect to \( \theta \) and so can be pulled out of
the expectation, and the expected value of \( \theta \) is \( \frac{1}{2} \). We know that for the
buyer to be willing to participate we must have

\[
p \leq \frac{v_b}{2}
\]

Now consider a seller of a type \( \theta \) is only willing to sell if

\[
v_s \theta \leq p
\]

or equivalently

\[
\theta \leq \frac{p}{v_s} \leq \frac{v_b}{2v_s}
\]

So if

\[
2v_s \leq v_b
\]

then it is possible to trade at price \( p \in [v_s, \frac{v_b}{2}] \) and get the efficient
outcome where trade always occurs.

What if \( 2v_s > v_b \)? Then we know that cars of quality higher than \( \frac{v_b}{2v_s} \)
would not be sold. Given that the car is offered for sale, the buyer must
know that the quality is less than \( \frac{v_b}{2v_s} \). Based on conditional probability
(don’t worry if you don’t know about conditional probability) the buyer
would then expect the quality to be uniformly distributed between 0 and
\( \frac{v_b}{2v_s} \). So then their expected valuation is

\[
E[v_b \theta \mid \theta < \frac{v_b}{2v_s}] = \frac{v_b^2}{4v_s}
\]
which must be at least as high as the price, so the seller will only be willing to participate if

$$\theta \leq \frac{P}{v_s} \leq \frac{v_b^2}{4v_s^2} = \left(\frac{v_b}{2v_s}\right)^2$$

and so the buyer must know that cars of quality above $\left(\frac{v_b}{2v_s}\right)^2$ will not be offered. If we continue this procedure $n$ times we see that cars of quality higher then $\left(\frac{v_b}{2v_s}\right)^n$ cannot be sold for any $n$. Taking $n \to +\infty$ we see that no cars (except possibly those of $\theta$ quality) can be sold in equilibrium whenever $v_b < 2v_s$.

What we see in the above example is that asymmetric information caused the complete collapse of the market, so that trade never occurs even though it is always efficient (the buyer always values the object more). Other examples with asymmetric information will not be so stark.

**Remark 33** If we instead assumed that neither the buyer or the seller know the realization of $\theta$ then the high quality cars would not be taken out of the market (sellers cannot condition their actions on information they do not have) and so we could have trade. This indicates that it is not the incompleteness of information that causes the problems, but the asymmetry.

**Remark 34** A slight variation of the above example indicates how price can convey information about quality in an asymmetric setting. Suppose $v_b > 2v_s$ and the buyer has the right to set the price (i.e. buyer makes a take it or leave it offer to the seller). If they set $p < v_s$ then only sellers of quality lower then $\frac{p}{v_s}$ will be willing to sell. So the expected quality of the car, given a willingness to sell at $p$ is $\frac{p}{2v_s}$ which is clearly increasing price. So the price reveals information about the expected quality.

### 11.2 Moral Hazard

Moral hazard is similar to asymmetric information except that instead of considering hidden information, it deals with hidden action. The distinction between the two concepts can be seen in an insurance example. Those who have pre-existing conditions that make them more risky (that are unknown to the insurer) are more likely, all else being equal, to buy insurance. This is adverse selection. An individual who has purchased insurance may become less cautious since the costs of any damage are covered by insurance company. This is moral hazard. There is a large literature in economics on how to structure incentives to mitigate moral
hazard. In the insurance example these incentives often take the form of deductibles and partial insurance, or the threat of higher premiums in response to accidents. Similarly an employer may structure a contract to include a bonus/commission rather than a fixed wage to enduce an employee to work hard. Below we consider an example of moral hazard, and show that a high price may signal an ability to commit to providing a high quality product.

Example 35 Suppose a cook can choose between producing a high quality meal ($q = 1$) and a low quality meal ($q = 0$). Assume that the cost of producing a high quality meal is strictly higher than a low quality meal ($c_1 > c_0$). For a meal of quality $i$, and price $p$ the benefit to the customer is $i - p$ and to the cook is $p - c_i$. So the total social welfare is

$$i - p + p - c_i = i - c_i$$

and assume that $1 - c_1 > 0 > -c_0$ so that the high quality meal is socially efficient. We assume that the price is set beforehand, and the cook’s choice variable is the quality of the meal. Obviously this is a limitation of the model that we are not considering how the price is set. Assume that fraction $\alpha$ of the consumers are repeat clients who are informed about the meal’s quality, whereas $1 - \alpha$ of the consumers are uniformed (visitors to the city perhaps) and don’t know the meal’s quality. The informed customers will only go to the restaurant if the meal is good (assume $p \in (0, 1)$). These informed customers allow us to consider a notion of reputation even though the model is static.

Now consider the decision of the cook as to what quality of meal to produce. If they produce a high quality meal then they sell to the entire market so their profits (per customer) are

$$p - c_1$$

Conversely, by producing the low quality meal, and selling to only $1 - \alpha$ of the market they earn profit

$$(1 - \alpha)(p - c_0)$$

and so the cook will provide the high quality meal if

$$p - c_1 \geq (1 - \alpha)(p - c_0)$$

or

$$-c_1 \geq -\alpha(p - c_0) - c_0$$
which corresponds to the case

$$\alpha \geq \frac{c_1 - c_0}{p - c_0}.$$ 

So the cook will provide the high quality meal if the fraction of the informed consumers is high enough. So informed consumers provide a positive externality on the uninformed, since the informed consumers will monitor the quality of the meal, inducing the chef to make a good meal.

Finally notice that price signals quality here: the higher the price the smaller the fraction of informed consumers necessary ensure the high quality meal. If the price is low \((p \approx c_1)\) then the cook knows he will lose \(p - c_1\) from each informed consumer by producing a low quality meal instead, but gains \(c_1 - c_0\) from each uninformed consumer (since the cost is lower). So only if almost every consumer is informed will the cook have an incentive to produce the good meal. As \(p\) increases so does \(p - c_1\), so the more is lost for each meal not sold to an informed consumer, and hence the lower the fraction of informed consumers necessary to ensure that the good meal will be provided. An uninformed consumer, who also may not know \(\alpha\), could then consider a high price a signal of high quality since it is more likely that the fraction of informed consumers is high enough to support the good meal the higher the price.