Useful Fact for Geometric Series

Fact. For $\delta \in (0, 1)$ and any constant $K$, $\sum_{t=0}^{\infty} K \delta^t = \frac{K}{1-\delta}$.

Why is the fact useful? When considering infinitely repeated games, we often have expressions that involve receiving a constant payoff $\pi$ each period: once today, again next period (discounted by $\delta$), again the following period, etc. With $\pi$ playing the role of $K$ above, the fact provides a simple expression for the value of the infinite sum.

Why is the fact true? (I will never ask you to show this, but you should want to see that this is true and the proof is not hard). For any finite integer $T > 0$, let

$$S_T = \sum_{t=0}^{T} \delta^t.$$ 

Then

$$(1-\delta) S_T = (1-\delta) \sum_{t=0}^{T} \delta^t$$

$$= \sum_{t=0}^{T} \delta^t - \sum_{t=0}^{T} \delta^{t+1}$$

$$= \sum_{t=0}^{T} \delta^t - \sum_{t=1}^{T+1} \delta^t$$

$$= \delta^0 - \delta^{T+1}$$

$$= 1 - \delta^{T+1}.$$ 

So

$$S_T = \frac{1 - \delta^{T+1}}{(1-\delta)}.$$ 

This is already pretty useful as it converts a possibly long sum into a single value. So this could be handy in a finitely repeated game, for example.

Now look at the limit of $S_T$ as $T \to \infty$ (which, after all, is what we mean by the value of an infinite sum). This is

$$\lim_{T \to \infty} S_T = \lim_{T \to \infty} \frac{1 - \delta^{T+1}}{(1-\delta)}$$

$$= \frac{1}{1-\delta}$$

since $\lim_{T \to \infty} \delta^{T+1} = 0$ (because $0 < \delta < 1$).
So, since $\sum_{t=0}^{\infty} \delta^t = \lim_{T \to \infty} S_T$,

$$\sum_{t=0}^{\infty} K \delta^t = K \sum_{t=0}^{\infty} \delta^t = \frac{K}{1 - \delta}.$$